Optimal Strategies
for Equal-Sum Dice Games

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Abstract

In this paper we consider a non-cooperative two-person zero-sum matrix game, called dice game. In an (n, σ) dice game, two players can independently choose a dice from a collection of hypothetical dice having n faces and with a total of σ eyes distributed over these faces. They independently roll their dice and the player showing the highest number of eyes wins (in case of a tie, none of the players wins). The problem at hand in this paper is the characterization of all optimal strategies for these games. More precisely, we determine the (n, σ) dice games for which optimal strategies exist and derive for these games the number of optimal strategies as well as their explicit form.

Key words: Dice game, Zero-sum matrix game, Non-cooperative game, Optimal strategy, Partitions.

1 Description of the dice game

The (n, σ) dice game is a game played between two players who want to obtain the highest individual profit. Both players choose independently a dice from the collection of (n, σ) dice. An (n, σ) dice is a fair, not necessarily materializable dice with n faces, each face containing a strictly positive number of eyes and the sum of the eyes on all faces being equal to the given σ. One can therefore also represent an (n, σ) dice by a partition of σ into n parts.

Definition 1 *The n-tuple $\pi = (i_1, i_2, \ldots, i_n)$ consisting of n strictly positive integers ordered non-decreasingly and with collective sum equal to σ, is called*

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a partition of σ into n parts. We will denote this type of partition by an (n, σ) partition.

Note that in partition theory the parts are usually ordered non-increasingly. In the sequel, we shall often identify an (n, σ) dice A_i with its unique associated partition π_i of σ into n parts.

Once the players have chosen their own dice (note that they might have selected the same one), the (n, σ) dice game is played in one or more rounds. At the beginning of each round, both players place a bet of €1 say, then independently roll their dice and compare the number of eyes on the bottom face: the dice that falls on the face with the highest number of eyes wins the round. The winner takes all and a new round can start. If the faces show the same number of eyes, the round ends in a draw: there is no winner, both players get their €1 back. Since the two players have as objective to win the game, and since each round of the game proceeds under the same conditions (same strategy, same bet), they want to choose from the collection a dice that maximizes their winning probability.

In order to compute the winning probability, let us regard dice A_i of player 1 and dice A_j of player 2 as independent discrete random variables, uniformly distributed on the multiset consisting of the number of eyes, for each face, of dice A_i and A_j, respectively. Recall that a multiset is a set that can contain the same number more than once. Therefore, a uniform distribution on a finite multiset is, in general, equivalent to a discrete distribution on an ordinary set endowed with a rational probability mass function. The winning probability q_{ij} of dice A_i w.r.t. dice A_j is the probability that the number on the bottom face of A_i is strictly greater than the number on the bottom face of A_j plus one half of the probability that both numbers are equal, or:

$$q_{ij} = \mathrm{Prob}\{A_i > A_j\} + \frac{1}{2}\,\mathrm{Prob}\{A_i = A_j\}\,. \tag{1}$$

Clearly, this definition implies that

$$q_{ij} + q_{ji} = 1\,. \tag{2}$$

Note that if both players have chosen the same dice, say A_i, then as they roll it independently, they obviously have the same winning probability $q_{ii} = 1/2$.

On the complete collection $\{A_i\}_{i=1}^{k}$ of (n, σ) dice, k denoting the number of partitions of σ into n parts, we consider the $[0, 1]$-valued relation $Q = [q_{ij}]$ consisting of the winning probabilities between all couples of dice, or equivalently, all couples of partitions. Due to property (2), Q is a so-called *reciprocal* relation, depending upon the context also known as an *ipsodual* or as a *probabilistic relation.*

Although the games considered are played with dice of the same type (same n and same σ), we will need to compare as well dice with same n but different σ. We therefore generalize (1), expressed immediately on the associated partitions. Consider an (n, σ_1) partition $\pi_1 = (i_1, i_2, \ldots, i_n)$ and an (n, σ_2) partition $\pi_2 = (j_1, j_2, \ldots, j_n)$, then we define

$$Q_{\pi_1,\pi_2} = \sum_{i_k > j_l} \frac{1}{n^2} + \frac{1}{2} \sum_{i_k = j_l} \frac{1}{n^2}\,. \tag{3}$$

Recently, we have studied reciprocal relations that are generated, more generally, by collections of arbitrary dice (not necessarily having the same number of faces or the same total sum of eyes and even not necessarily fair). In particular, we have been able to characterize the transitivity of the generated reciprocal relations in the framework of cycle-transitivity [2,5]. This framework largely generalizes stochastic transitivity [6]. We have further generalized the dice concept to establish a model that can be used for the pairwise comparison of arbitrary discrete or continuous, independent or dependent distributions [3,4] and that provides some interesting alternatives to the classical notion of stochastic dominance of distributions, widely used in financial mathematics, welfare models and risk theory [7]. In the present paper, we stick to discrete distributions on integer multisets of same cardinality n and with sum of the integers equal to σ, for otherwise the resulting game would be a trivial one. For a given σ, for instance, when n can be freely chosen, the optimal choice is $n = 1$, resulting in a hypothetical dice with 1 face that contains σ eyes.

In the next section, we characterize the dice game described above in the formal setting of game theory. Section 3 gives in the form of a theorem and a number of propositions and corollaries a clear answer to the following questions: for which values of n and σ do there exist optimal dice, and if such dice exist, how many are they and what is their precise form? Section 4 contains the proof of the main results covered in Section 3. To make these proofs as comprehensible as possible, examples of (n, σ) dice games and their strategies will be used at different places to illustrate the theoretical results.

2 Game-theoretic characterization of the (n, σ) dice game

Since both players in an (n, σ) dice game want to optimize their winning chances, the game belongs to the class of *non-cooperative games*. For game-theoretic terminology, see e.g. [8,9]. The (n, σ) dice A_i, with $i = 1, 2, \ldots, k$, are called *pure strategies*. Let us denote the set of all pure strategies as A. The problem of finding the best dice therefore amounts to finding the *optimal* *strategies* of the game. In this respect, we define the *payoff function* $p^{(1)}$: $A \times A \to [-1/2, 1/2]$ of player 1 by:

$$p^{(1)}(A_i, A_j) = p_{ij}^{(1)} = q_{ij} - \frac{1}{2}, \tag{4}$$

where the first argument A_i denotes the strategy of the first player and the second argument A_j the strategy of the second player. It follows that the payoff function $p^{(2)}$ of player 2 is then given by:

$$p^{(2)}(A_i, A_j) = p_{ij}^{(2)} = (1 - q_{ij}) - \frac{1}{2} = -p_{ij}^{(1)}, \tag{5}$$

where the meaning of the two arguments is the same as in (4). Note that the payoff $2p_{ij}^{(d)}$ lies in the interval $[-1, 1]$ and is for $d = 1, 2$ nothing else than the expected gain (expressed in €) of player d in a single round (when both players bet € 1). The fact that the sum of the values of the payoff function of both players equals zero for each situation allows us to identify the dice game as an *antagonistic game*, or in other words a zero-sum two-person game. As it also holds that $p_{ij}^{(1)} = -p_{ji}^{(1)}$, the game is a *symmetric game*.

Antagonistic games in which each player has a finite number of strategies, which is clearly the case here, are also called *matrix games*. A matrix game is completely determined by its *payoff matrix*, which is given by $P = [p_{ij}^{(1)}]$. As an example, the payoff matrix for the $(6, 12)$ dice game is given in Figure 1.

A situation is called *admissible* for a player if by replacing her present strategy in this situation by some other strategy, the player is unable to increase the payoff. When the situation is admissible for both players, this situation is called a *saddle point* and the strategies are called *optimal strategies*. For a symmetric game, it holds that the payoff in a saddle point equals 0. For a player to maximize her winning probability, she needs to choose a dice that is an optimal strategy, if there is one. If she chooses this dice, she is assured that the probability that she wins is greater than or equal to 1/2, no matter which dice the other player chooses. If on the other hand, she does not choose an optimal strategy, but the other player does, she is assured that her winning probability is less than or equal to 1/2.

In the payoff matrix of Figure 1, the saddle points are encircled. As can be seen from this example payoff matrix, there are 4 optimal strategies and therefore 16 saddle points. It can be verified that for each row containing a saddle point the payoffs are greater than or equal to 0 and for each column containing a saddle point they are less than or equal to zero. For any situation that is not a saddle point, one can either find a situation on the same row that has a smaller payoff, or a situation in the same column that has a bigger payoff.

The rest of this paper is devoted to the characterization of all optimal strategies of the (n, σ) dice games. It must be noted, however, that not all (n, σ) dice games have optimal strategies and for these games we obviously cannot state the strategies a player should pick to maximize her winning probabilities.

We end this section with the introduction of two notations related to the theory of partitions [1], which we will frequently use for illustrating and proving our main results.

It is sometimes helpful to use a notation that makes explicit the number of times a particular integer appears in a partition. We use the same notation as in partition theory and call this the tally-representation of the partition.

$$\frac{1}{72} \times \begin{pmatrix} 0 & -4 & -4 & -4 & -9 & -9 & -9 & -14 & -14 & -19 & -24 \\ 4 & 0 & 0 & 0 & -4 & -5 & -6 & -9 & -10 & -14 & -18 \\ 4 & 0 & 0 & 0 & -3 & -3 & -3 & -6 & -6 & -9 & -12 \\ 4 & 0 & 0 & 0 & -4 & -2 & 0 & -6 & -4 & -8 & -12 \\ 9 & 4 & 3 & 4 & 0 & -1 & -3 & -4 & -6 & -9 & -12 \\ 9 & 5 & 3 & 2 & 1 & 0 & 0 & -2 & -2 & -4 & -6 \\ 9 & 6 & 3 & 0 & 3 & 0 & \textcircled{0} & 0 & \textcircled{0} & \textcircled{0} & \textcircled{0} \\ 14 & 9 & 6 & 6 & 4 & 2 & 0 & 0 & -2 & -4 & -6 \\ 14 & 10 & 6 & 4 & 6 & 2 & \textcircled{0} & 2 & \textcircled{0} & \textcircled{0} & \textcircled{0} \\ 19 & 14 & 9 & 8 & 9 & 4 & \textcircled{0} & 4 & \textcircled{0} & \textcircled{0} & \textcircled{0} \\ 24 & 18 & 12 & 12 & 12 & 6 & \textcircled{0} & 6 & \textcircled{0} & \textcircled{0} & \textcircled{0} \end{pmatrix}$$

Fig. 1. Payoff matrix for the $(6, 12)$ dice game

Definition 2 *The tally-representation of an (n, σ) partition $\pi = (i_1, i_2, \ldots, i_n)$ is given by $(1^{t_1} 2^{t_2} 3^{t_3} \ldots)$ in which t_i denotes the number of times i appears in the partition. When $t_i = 0$ the entry i^{t_i} can be omitted.*

For the tally-representation $(1^{t_1} 2^{t_2} 3^{t_3} \ldots)$ of a given (n, σ) partition π it clearly holds that $0 \leq t_i \leq n$, $\sum_{i>0} t_i = n$ and $\sum_{i>0} i\, t_i = \sigma$.

In some proofs, we will use the concepts of decremented and incremented partitions.

Definition 3

(1) The decremented partition $\delta(\pi, m)$ corresponding to a given (n, σ) partition $\pi = (i_1, i_2, \ldots, i_n)$ is the $(n, \sigma - 1)$ partition obtained by decrementing the element i_m of π, $1 \leq m \leq n$, where it is assumed that $i_m \neq 1$.

(2) The incremented partition $\nu(\pi, m)$ corresponding to a given (n, σ) partition $\pi = (i_1, i_2, \ldots, i_n)$ is the $(n, \sigma+1)$ partition obtained by incrementing the element i_m of π, $1 \le m \le n$.

3 Optimal strategies for (n, σ) dice games

In this section, we merely state the main results characterizing the optimal strategies for (n, σ) dice games, whereas the proofs of the theorems and propositions will be the concern of the next section.

A first important observation is that not all (n, σ) dice games possess one or more optimal strategies. The following theorem formulates the necessary and sufficient conditions for an (n, σ) dice game to have at least one optimal strategy.

Theorem 4 *An (n, σ) dice game has at least one optimal strategy if and only if one of the following six mutually exclusive conditions is satisfied:*

(i) $n \le 2$
(ii) $(n, \sigma) = (3, 7)$
(iii) $(n, \sigma) = (3, 8)$
(iv) $(n, \sigma) = (2l, 4l + 1)\,, l > 1$
(v) $n > 2$ *and there exist* $a, b, k \in \mathbb{N}$ *such that*

$$\begin{cases} n = (a + b)\,k - b \\ \sigma = n\,k \end{cases} \tag{6}$$

(vi) $n > 2$ *and there exist* $a, b, k \in \mathbb{N}$ *such that*

$$\begin{cases} n = (a + b)\,k \\ \sigma = (n + b)\,k \\ a \neq 0 \wedge b \neq 0 \end{cases} \tag{7}$$

The mutual exclusivity of the above 6 conditions is a matter of direct verification. While the above theorem characterizes all (n, σ) dice games that possess optimal strategies, the propositions below state the number of optimal strategies and their explicit form. We start by handling the special cases.

Proposition 5

(1) The $(1, \sigma)$ dice game: the unique strategy (σ^1) is optimal.

(2) *The* $(2,\sigma)$ *dice game: all* $\lfloor \frac{\sigma}{2} \rfloor$ *strategies* $(k^1(\sigma-k)^1)$, $0 < k \leq \lfloor \frac{\sigma}{2} \rfloor$, *are optimal.*
(3) *The* $(3,7)$ *dice game:* $(1^1 3^2)$ *is the only optimal strategy.*
(4) *The* $(3,8)$ *dice game:* $(1^1 3^1 4^1)$ *is the only optimal strategy.*
(5) *The* (n,n) *dice game: the unique strategy* (1^n) *is optimal.*
(6) *The* $(2n,4n{+}1)$ *dice game,* $n>1$*:* $(1^{n-1}2^1 3^n)$ *is the only optimal strategy.*

The next proposition discusses the dice games of type (6), excluding the (n,n) dice game which has already been considered in the above proposition.

Proposition 6 *All* (n,σ) *dice games, with* $n \neq \sigma$*, satisfying* (6) *have exactly* $\lfloor a/(k-1) \rfloor + \lfloor b/k \rfloor + 1$ *optimal strategies and their tally-representation is given by* $(1^a 2^b 3^a 4^b \ldots (2k-2)^b (2k-1)^a)$*, where* a, b *are different but* k *is the same for each optimal strategy.*

Example 7 The $(6,12)$ dice game, for which the payoff matrix is given in Figure 1, has the following pure strategies:

$$\begin{array}{lll}
\pi_1 = (1,1,1,1,1,7) & \pi_5 = (1,1,1,2,2,5) & \pi_9 \ = (1,1,2,2,3,3) \\
\pi_2 = (1,1,1,1,2,6) & \pi_6 = (1,1,1,2,3,4) & \pi_{10} = (1,2,2,2,2,3) \\
\pi_3 = (1,1,1,1,3,5) & \pi_7 = (1,1,1,3,3,3) & \pi_{11} = (2,2,2,2,2,2) \\
\pi_4 = (1,1,1,1,4,4) & \pi_8 = (1,1,2,2,2,4) &
\end{array}$$

With $n=6$ and $\sigma=12$, system (6) has the following solutions ($k=2$): $a=3$ and $b=0$, $a=2$ and $b=2$, $a=1$ and $b=4$, $a=0$ and $b=6$. According to Proposition 6, the tally-representations of the corresponding optimal strategies are given by:

(i) $a=3$ and $b=0$: $(1^3 3^3)$
(ii) $a=2$ and $b=2$: $(1^2 2^2 3^2)$
(iii) $a=1$ and $b=4$: $(1^1 2^4 3^1)$
(iv) $a=0$ and $b=6$: (2^6)

These clearly correspond to the partitions π_7, π_9, π_{10} and π_{11}. One can verify that for any of the above a and b it indeed holds that $\lfloor \frac{a}{k-1} \rfloor + \lfloor \frac{b}{k} \rfloor + 1 = 4$. ▷

Finally, the games of type (7) are considered.

Proposition 8 *All* (n,σ) *dice games satisfying* (7) *have exactly one optimal strategy* $(1^a 2^b 3^a 4^b \ldots (2k-1)^a (2k)^b)$.

Example 9

(i) The $(6,21)$ dice game has 110 strategies and satisfies (7), with $k=3$ and $a=b=1$. The unique optimal strategy for this game is given by

(1, 2, 3, 4, 5, 6), the usual dice.

(ii) The (8, 22) dice game has 116 strategies and satisfies (7), with $k = 2$, $a = 1$ and $b = 3$. The unique optimal strategy for this game is given by $(1^1 2^3 3^1 4^3)$. ▷

The above propositions imply the following corollaries, which are statements about certain types of diophantine systems. From Proposition 6 it follows that:

Corollary 10 *For given values of n and σ, $n \neq \sigma$, the entity $\lfloor a/(k-1) \rfloor + \lfloor b/k \rfloor$ is an invariant of the solution space of system* (6)*. If this system has a solution, then it has exactly $\lfloor a/(k-1) \rfloor + \lfloor b/k \rfloor + 1$ solutions.*

On the other hand, Proposition 8 implies:

Corollary 11 *For given values of n and σ, system* (7) *has at most one solution.*

4 Proof of the main results

The proof of the theorem and propositions from Section 3 is realized by dividing the collection of (n, σ) partitions in different classes and by determining for each class separately the partitions that are optimal, if there are any. With each class of partitions we will moreover associate one or more subcases.

Cases 1–5 cover the (n, σ) dice games specified in items (*i*)–(*iii*) from Theorem 4 and items 1–5 from Proposition 5. After considering these cases, we will introduce the increment/decrement operation, which will play a central role when considering the subsequent cases. Case 6 considers a specific type of strategies for which it is proven that they cannot be optimal. Case 7 considers another class of strategies for which it is shown that there exists a limited subset of optimal strategies. In this subset, the only strategies not yet covered in previous cases, are those stated in item 6 of Proposition 5. The proof of item (*iv*) in Theorem 4 is then immediate. Finally, case 8 considers all remaining strategies, not yet covered by the previous cases. These strategies can be nicely characterized and we will divide them in three subclasses, each leading to a subcase in the proof. Subcases 8.1 and 8.2 are concerned with subsets of strategies that will be proven to be non-optimal, while in subcase 8.3 we consider the remaining strategies, which are shown to be the optimal strategies mentioned in Propositions 6 and 8. First, we will prove that these strategies are indeed optimal, then we will prove that they only exist when either condition (6) or condition (7) is satisfied. Also, the number of these optimal strategies will be counted in order to finalize the proof of Propositions 6 and 8. Corollaries 10 and 11 will immediately follow from these results.

4.1 Proofs for some special (n, σ) dice games

As mentioned above, we first consider some special cases that cannot be handled in a more general way. These all have optimal strategies.

Case 1: The $(1, \sigma)$ dice game:

There is only one strategy in this type of game, namely (σ^1), and this strategy is therefore optimal.

Case 2: The $(2, \sigma)$ dice game:

For any two $(2, \sigma)$ partitions π_1 and π_2, it holds that $Q_{\pi_1,\pi_2} = 1/2$. Indeed, for any two distinct $(2, \sigma)$ partitions $\pi_1 = (a_1, a_2)$ and $\pi_2 = (b_1, b_2)$, we have that either $a_1 < b_1 \leq b_2 < a_2$ or $b_1 < a_1 \leq a_2 < b_2$, from which it follows that $Q_{\pi_1,\pi_2} = 1/2$. Therefore, any $(2, \sigma)$ partition is an optimal strategy. Moreover, it is obvious that there exist exactly $\lfloor \frac{\sigma}{2} \rfloor$ such $(2, \sigma)$ partitions.

Case 3: The $(3, 7)$ dice game:

In the $(3, 7)$ dice game, it holds that $(1, 3, 3)$ is the only optimal strategy. Indeed, there are four $(3, 7)$ partitions, namely: $(1, 1, 5)$, $(1, 2, 4)$, $(1, 3, 3)$, $(2, 2, 3)$. Easy calculations support the stated result.

Case 4: The $(3, 8)$ dice game:

In the $(3, 8)$ dice game, it holds that $(1, 3, 4)$ is the only optimal strategy. Indeed, there are five $(3, 8)$ partitions: $(1, 1, 6)$, $(1, 2, 5)$, $(1, 3, 4)$, $(2, 2, 4)$, $(2, 3, 3)$. Again, easy calculations support the stated result.

Case 5: The (n, n) dice game:

There is only one strategy in this dice game, namely (1^n), which is therefore optimal.

4.2 Decremented and incremented partitions reconsidered

In what follows, we can exclude the above special cases. Before going further, we need to introduce some concepts related to decremented partitions.

Consider an (n, σ) partition $\pi = (i_1, i_2, \ldots, i_n)$. Let m, $1 \leq m \leq n$, be such that $i_m \neq 1$ and

$$(\forall\ 1 \leq j \leq n)\,(i_j \neq 1 \Rightarrow Q_{\pi,\delta(\pi,j)} \leq Q_{\pi,\delta(\pi,m)})\,. \tag{8}$$

Such values of m will be called *max-decrement positions*. Property (8) specifies that the probability that the original partition π wins from the decremented partition $\delta(\pi, i)$ is highest when i is a max-decrement position. Note that such a value is not necessarily unique. For example, the $(10, 39)$ partition $\pi = (1, 1, 1, 3, 3, 4, 6, 6, 6, 8)$ has 6, 7, 8 or 9 as possible max-decrement positions, where we have that $\delta(\pi, 6) = (1, 1, 1, 3, 3, 3, 6, 6, 6, 8)$, $\delta(\pi, 7) = \delta(\pi, 8) = \delta(\pi_1, 9) = (1, 1, 1, 3, 3, 4, 5, 6, 6, 8)$ and $Q_{\pi,\delta(\pi,6)} = Q_{\pi,\delta(\pi,7)} = 1/2 + 3/200$.

Since (n, n) dice games are excluded, at least one such max-decrement position exists. Clearly, for any $i_k \neq 1$ it holds that

$$Q_{\pi,\delta(\pi,k)} = Q_{\pi,\pi} + \frac{t_{i_k} + t_{i_k-1}}{2n^2} = \frac{1}{2} + \frac{t_{i_k} + t_{i_k-1}}{2n^2} .$$

Also, incrementing i_k in π gives rise to the following equality:

$$Q_{\pi,\nu(\pi,k)} = \frac{1}{2} - \frac{t_{i_k} + t_{i_k+1}}{2n^2} .$$

Next, suppose we increment i_k and decrement i_l, $i_l \neq 1$ and $l \neq k$, in an (n, σ) partition π_1, then we obtain an (n, σ) partition π_2. We call this operation an increment/decrement operation. The following equality holds:

$$Q_{\pi_1,\pi_2} = \frac{1}{2} + \frac{t_{i_l} + t_{i_l-1} - t_{i_k} - t_{i_k+1}}{2n^2} .$$

In the next cases, when proving that a given partition π_1 is not an optimal strategy, we will construct a partition π_2 such that $Q_{\pi_1,\pi_2} < 1/2$ by means of increment/decrement operations. Obviously, this construction of a partition π_2 that wins in overall from partition π_1 is in general not unique.

4.3 The proof continued: the $(2l, 4l + 1)$ dice games

We now divide the set of all remaining (n, σ) partitions into three classes and investigate each class separately.

Case 6: Consider an (n, σ) partition $\pi_1 = (1^{t_1} 2^{t_2} \ldots)$ such that

$$(\exists\, j > 0)\ (t_j = 0 \wedge t_{j+1} = 0 \wedge t_{j+2} \neq 0) . \tag{9}$$

Decrement an occurrence of $j + 2$ by 2 and increment two different elements l (if $t_{j+2} > 1$ then choose l to be another occurrence of $j + 2$) and m from partition π_1. The resulting partition π_2 wins from π_1. Indeed, it holds that

$$Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_1} - \frac{t_l + t_{l+1} + t_m + t_{m+1} - t_{j+2}}{2n^2} .$$

From the right-hand side it is seen that Q_{π_1,π_2} is strictly smaller than $1/2$, since it clearly holds that $t_l > 0$ and $t_m > 0$ and that $l = j+2$ when $t_{j+2} > 1$.

Example 12

(i) Consider the $(4, 15)$ partition $\pi_1 = (1, 4, 4, 6)$ then formula (9) is satisfied for $j = 2$. For the $(4, 15)$ partition $\pi_2 = (2, 2, 5, 6)$ it holds that $Q_{\pi_1,\pi_2} = \frac{15}{32} < \frac{1}{2}$.

(ii) Consider the $(3, 12)$ partition $\pi_1 = (3, 4, 5)$ then formula (9) is satisfied for $j = 1$. For $\pi_2 = (1, 5, 6)$ it holds that $Q_{\pi_1,\pi_2} = \frac{7}{18} < \frac{1}{2}$. ▷

Case 7: Consider an (n, σ) partition $\pi_1 = (1^{t_1} 2^{t_2} \ldots)$ such that

$$(\exists\ m',\ i_{m'} \neq 1)\ (Q_{\pi_1,\delta(\pi_1,m')} < Q_{\pi_1,\delta(\pi_1,m)})\,, \tag{10}$$

where m is a max-decrement position as defined in (8). We can safely assume that (9) does not hold as that case was covered before.

First assume there exists an m satisfying (8) and for which $t_{i_m-1} \neq 0$, together with an m' satisfying (10). Furthermore, assume that it holds that $i_{m'} \neq i_m - 1$ or $t_{i_{m'}} > 1$. We need at least one of these two conditions to hold because otherwise it is impossible to increment an occurrence of $i_m - 1$ and decrement an occurrence of $i_{m'}$. So, we are able to construct π_2 starting from π_1 by incrementing an occurrence of $i_m - 1$ and decrementing $i_{m'}$. Noting that $t_{i_m-1} + t_{i_m} > t_{i_{m'}-1} + t_{i_{m'}}$, we obtain due to (10) that:

$$Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_1} - \frac{t_{i_m-1} + t_{i_m} - t_{i_{m'}} - t_{i_{m'}-1}}{2n^2} < \frac{1}{2}\,.$$

Secondly, let us assume there exists an m satisfying (8) and for which $t_{i_m-1} = 0$, together with an m' satisfying (10). We build π_2 starting from π_1, by incrementing i_m and decrementing $i_{m'}$. Noting that $t_{i_m} > t_{i_{m'}-1} + t_{i_{m'}}$, we now obtain:

$$Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_1} - \frac{t_{i_m} - t_{i_{m'}} - t_{i_{m'}-1}}{2n^2} < \frac{1}{2}\,.$$

Note that t_{i_m+1} must equal 0, as m is a max-decrement position and $t_{i_m-1} = 0$, and therefore we can safely omit it in the above expression.

It is easy to see that the only partitions not covered by the previous assumptions while satisfying (10) and not satisfying (9), correspond to the following two types of partitions:

$$\pi_1 = (1^{t_1} 3^1 4^{t_4}) \quad \text{, with } t_1 > 0 \text{ and } t_4 > 0\,, \tag{11}$$
$$\pi_1 = (1^{t_1} 2^1 3^{t_3}) \quad \text{, with } 0 \leq t_1 < t_3\,. \tag{12}$$

Indeed, it must hold that $i_{m'}$ and i_m are unique, that $i_{m'} = i_m - 1$, $t_{i_{m'}} = 1$ and that (9) is not satisfied. There must only be one possible choice for $i_{m'}$ and i_m

as there otherwise would exist a choice such that $i_{m'} \neq i_m - 1$. The fact that i_m and $i_{m'}$ are unique implies that $\#\{i \mid i > 1 \wedge t_i \neq 0\} = 2$. Furthermore, the fact that $i_m - 1 = i_{m'}$ and that (9) is not satisfied, imply that either $m = 4$ and $t_1 \neq 0$, or $m = 3$ and $t_1 < t_3$.

First, consider partitions of type (11). If $t_1 > 1$, then we make π_2 from π_1 by incrementing an occurrence of 1 and decrementing 3. We obtain:

$$Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_1} - \frac{t_1 - 1}{2n^2} < \frac{1}{2}\,.$$

Now, suppose $t_1 = 1$. Unless there is only one occurrence of 4, which corresponds to Case 4, we can construct π_2 from π_1 by decrementing 3 and incrementing an occurrence of 4. We obtain:

$$Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_1} - \frac{t_4 - 1}{2n^2} < \frac{1}{2}\,.$$

Secondly, let us consider partitions of type (12). When $t_1 < t_3 - 1$, the partition $\pi_2 = (1^{t_1+1}3^{t_3-1}4^1)$ wins from π_1, since:

$$Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_1} - \frac{t_3 - t_1 - 1}{2n^2} < \frac{1}{2}\,.$$

Suppose now that $t_1 = t_3 - 1$. If $t_1 = 0$, then $\pi_1 = (2, 3)$ belongs to the class of partitions covered in Case 2. If $t_1 > 0$, then the partition π_1 is of type $(1^{l-1}2^1 3^l)$, with $l > 1$, and these are all optimal strategies. Indeed, using increment/decrement operations, we can transform the $(2l, 4l + 1)$ partition $\pi_1 = (1^{l-1}2^1 3^l)$ into any other $(2l, 4l + 1)$ partition. First note that an increment of 2 is useless, as the number to be decremented is then at least 3, which means an earlier increment/decrement operation would be cancelled out. Therefore, for any partition π_1', obtained as an intermediate step in this increment/decrement process, it holds that an increment causes a decrease of at most $l/(2n^2)$, while a decrement leads to an increase of at least $l/(2n^2)$. Hence, no increment/decrement operation can lead to a decrease and $Q_{\pi_1,\pi_2} \geq 1/2$, for any $(2l, 4l + 1)$ partition π_2. As it is easily verified that for $(n, \sigma) = (2l, 4l + 1)$ the diophantine systems (6) and (7) have no solution, it follows from Theorem 4 (of which a part of the proof still needs to be given below), that $(1^{l-1}2^1 3^l)$ is the only optimal strategy of the $(2l, 4l + 1)$-game, with $l > 1$.

Example 13

(i) Consider the $(6, 23)$ partition $\pi_1 = (1, 2, 3, 5, 6, 6)$, $m = 5$ or $m = 6$, $t_{i_m - 1} = 1$, then the possible values for m' are 2 and 3. Choosing π_2 to be one of the partitions $(1, 1, 3, 6, 6, 6)$ and $(1, 2, 2, 6, 6, 6)$, we obtain $Q_{\pi_1,\pi_2} = \frac{35}{72}$.

(ii) Consider the $(4, 11)$ partition $\pi_1 = (1, 3, 3, 4)$, $m = 4$, $i_{m'} = i_m - 1$, $t_{i_{m'}} = 2 > 1$. If we choose $\pi_2 = (1, 2, 4, 4)$, then $Q_{\pi_1,\pi_2} = \frac{15}{32}$.

(iii) Consider the $(4, 12)$ partition $\pi_1 = (1, 3, 4, 4)$. For $\pi_2 = (1, 2, 4, 5)$, we find $Q_{\pi_1,\pi_2} = \frac{15}{32}$.

(iv) Consider the $(3, 8)$ partition $\pi_1 = (2, 3, 3)$. If we choose $\pi_2 = (1, 3, 4)$, then $Q_{\pi_1,\pi_2} = \frac{4}{9}$. ▷

4.4 The proof continued: investigation of the remaining dice games

Case 8: If (9) and (10) are not satisfied, then the partition π_1 should satisfy the following property, for some fixed $C \in \mathbb{N}_0$:

$$\begin{cases} i > 1 \wedge t_i > 0 \Rightarrow t_{i-1} + t_i = C\,, \\ (\forall\, i < i_n)(t_i + t_{i+1} > 0)\,. \end{cases} \tag{13}$$

The first property holds because (10) is not satisfied, the second property because (9) is not satisfied.

The remaining cases for π_1 are therefore of one of the following types ($a, b \in \mathbb{N}$, $k, k' \in \mathbb{N}_0$, $k < k'$):

$$(1^a 2^b \ldots (2k)^b (2k+1)^a (2k+3)^{a+b} (2k+5)^{a+b} \ldots (2k'+1)^{a+b}) \tag{14}$$

$$(1^a 2^b \ldots (2k-1)^a (2k)^b (2k+2)^{a+b} (2k+4)^{a+b} \ldots (2k')^{a+b}) \tag{15}$$

$$(1^a 2^b \ldots (2k-2)^b (2k-1)^a) \tag{16}$$

$$(1^a 2^b \ldots (2k-1)^a (2k)^b) \tag{17}$$

$$(1^a 3^b 5^b \ldots (2k+1)^b) \tag{18}$$

To assure that these five cases are mutually exclusive, the following conditions on a and b must be imposed. For type (14) and (15), $a \neq 0$ and $b \neq 0$ must hold because else π_1 would correspond to type (16) or (17), or (9) would hold. For type (17) it must hold that $a \neq 0$ and $b \neq 0$ (making (16) and (17) mutually exclusive). For type (18) it should hold that $a \neq b$, $a \neq 0$ and $b \neq 0$ in order to make it mutally exclusive with (16) and to exclude the partitions considered already in Cases 5 and 6.

Subcase 8.1: Suppose π_1 is of type (14) or (15), with $a \neq 0$ and $b \neq 0$.

Let $\nu = \min\{i \mid t_i = 0\}$. Clearly $\nu > 2$ and $t_{\nu+1} > 1$. Decrement an occurrence of $\nu + 1$ by 2, increment another occurrence of $\nu + 1$ by 1 and increment an occurrence of 1 by one. The resulting partition is the (n, σ) partition π_2, which clearly wins from π_1: for case (14) we obtain

$$Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_1} - \frac{b}{2n^2} < 1/2\,,$$

while for case (15) we obtain

$$Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_1} - \frac{a}{2n^2} < 1/2\,.$$

Example 14

(i) Consider the $(12, 39)$ partition $\pi_1 = (1^2 2^3 3^2 5^5)$. For the partition $\pi_2 = (1^1 2^4 3^3 5^3 6^1)$, it holds that $Q_{\pi_1,\pi_2} = \frac{141}{288}$.

(ii) Consider the $(12, 58)$ partition $\pi_1 = (1^1 2^2 3^1 4^2 6^3 8^3)$. For the partition $\pi_2 = (2^3 3^1 4^3 6^1 7^1 8^3)$, we find $Q_{\pi_1,\pi_2} = \frac{143}{288}$. ▷

Subcase 8.2: Suppose π_1 is of type (18), with $a \neq b$, $a \neq 0$ and $b \neq 0$.

Let us first consider $a > b$. We construct π_2 from π_1 by incrementing an occurrence of 1 and decrementing an occurrence of 3 to obtain:

$$Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_1} - \frac{a-b}{2n^2} < \frac{1}{2}\,.$$

Next, suppose $a < b$. If $b > 2$, then we construct π_2 from π_1 by decrementing an occurrence of 3 by two and incrementing two other occurrences of 3 by one. We obtain:

$$Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_1} - \frac{b-a}{2n^2} < \frac{1}{2}\,.$$

When $0 < a < b = 2$ and $n > 3$, we construct π_2 from π_1 by decrementing an occurrence of 3 by two, incrementing the other occurrence of 3 by one and incrementing an occurrence of 5 by one. We obtain:

$$Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_1} - \frac{b-a}{2n^2} < \frac{1}{2}\,.$$

The case $a = 1$, $b = 2$ and $n = 3$ corresponds to Case 3.

Example 15

(i) Consider the $(7, 19)$ partition $\pi_1 = (1^3 3^2 5^2)$. If we choose $\pi_2 = (1^2 2^2 3^1 5^2)$, then $Q_{\pi_1,\pi_2} = \frac{24}{49}$.

(ii) Consider the $(4, 10)$ partition $\pi_1 = (1^1 3^3)$. For $\pi_2 = (1^2 4^2)$, it holds that $Q_{\pi_1,\pi_2} = \frac{7}{16}$.

(iii) Consider the $(5, 17)$ partition $\pi_1 = (1^1 3^2 5^2)$. Choosing $\pi_2 = (1^2 4^1 5^1 6^1)$, it follows that $Q_{\pi_1,\pi_2} = \frac{12}{25}$. ▷

Subcase 8.3: Suppose π_1 is of type (16) or (17) (with $a \neq 0$ and $b \neq 0$ in case of (17)), which implies:

$$(\forall\, i < i_n)(t_i + t_{i+1} = C = a + b)\,. \tag{19}$$

We will prove that such (n, σ) partitions do not lose from any other (n, σ) partition. Consider an (n, σ) partition $\pi_2 = (j_1, \ldots, j_n) = (1^{t'_1} 2^{t'_2} \ldots)$. As π_1 is also an (n, σ) partition, we can obtain π_2 from π_1 step by step using increment/decrement operations on the elements of the intermediate partitions.

When $j_n \leq i_n$ we can obtain π_2 from π_1 gradually by repeatedly incrementing some $k \in \pi_i$ with $k < i_n$ in the intermediate partition π_i and decrementing another l in π_i until partition π_2 is obtained. It is obvious that after every increment/decrement operation, obtaining an intermediate partition π_i, it holds that $Q_{\pi_1,\pi_i} = Q_{\pi_1,\pi_1}$. Indeed, consider such an intermediate partition π_i and let k (resp. l) be the number to be incremented (resp. decremented). From (19) it follows that $t_{l-1} + t_l = C = t_k + t_{k+1}$ (recall that $k < \mu$). Let π'_i be the partition obtained from π_i after the mentioned increment/decrement operation. We obtain:

$$Q_{\pi_1,\pi'_i} = Q_{\pi_1,\pi_i} + \frac{t_l + t_{l-1} - t_k - t_{k+1}}{2n^2} = Q_{\pi_1,\pi_i}\,.$$

Since the end result of the transformation is π_2 and we started from π_1, we obtain $Q_{\pi_1,\pi_2} = Q_{\pi_1,\pi_i} = Q_{\pi_1,\pi_1} = 1/2$.

When $j_n > i_n$ we can use increment/decrement operations to obtain π_2 from π_1 by only decrementing numbers $l \leq i_n$. There will be at least one increment/decrement operation that decrements a number $l \leq i_n$ and increments $k = i_n$. Therefore, it holds for case (16) that $Q_{\pi_1,\pi_2} \geq Q_{\pi_1,\pi_1} + \frac{b}{2n^2}$ and for case (17) that $Q_{\pi_1,\pi_2} \geq Q_{\pi_1,\pi_1} + \frac{a}{2n^2}$. This proves that the (n, σ) partition π_1 does not lose from any (n, σ) partition and therefore π_1 is an optimal strategy.

Example 16

(i) Consider the $(12, 36)$ partition $\pi_1 = (1^2 2^3 3^2 4^3 5^2)$, which is of type (16), and the $(12, 36)$ partition $\pi_2 = (1^2 2^1 3^4 4^5)$. Using increment/decrement operations, we can transform π_1 into π_2:

$$\pi_1 = (1^2 2^3 3^2 4^3 5^2) \to \pi'_1 = (1^2 2^2 3^3 4^4 5^1) \to \pi''_1 = \pi_2 = (1^2 2^1 3^4 4^5)\,.$$

It holds that $Q_{\pi_1,\pi_1} = 1/2 = Q_{\pi_1,\pi'_1} = Q_{\pi_1,\pi''_1} = Q_{\pi_1,\pi_2}$.

(ii) Consider the $(10, 26)$ partition $\pi_1 = (1^2 2^3 3^2 4^3)$, which is of type (17), and the $(10, 26)$ partition $\pi_2 = (1^3 2^4 5^3)$. We again transform π_1 into π_2:

$$\pi_1 = (1^2 2^3 3^2 4^3) \to \pi'_1 = (1^3 2^2 3^2 4^2 5^1) \to \pi''_1 = (1^3 2^3 3^1 4^1 5^2)$$
$$\to \pi'''_1 = \pi_2 = (1^3 2^4 5^3)\,.$$

Since $Q_{\pi_1,\pi_1'} = \frac{51}{100} > 1/2$, we obtain $Q_{\pi_1,\pi_2} > 1/2$. ▷

All possible (n, σ) partitions have been considered in the above cases and the previously obtained results already show how the optimal strategies look. We still need to prove, for (n, σ) games with $n > 2$, that the existence of partitions of type (16), resp. (17), is equivalent to condition (6), resp. (7), from Theorem 4 and obtain the number of optimal strategies for (n, σ)-games of one of these two types. As was already mentioned, the first three conditions of Theorem 4 correspond to the special Cases 1–4 considered in this subsection. The fourth condition of Theorem 4 was obtained in Case 7.

4.5 Finalizing the proof of Theorem 4

We will now prove, for $n > 2$, that condition (6), resp. (7), is equivalent to the condition that there exists at least one (n, σ) partition π_1 that is of type (16), resp. (17), where for the case of (17) it is required that $a \neq 0$ and $b \neq 0$. It is obvious that case (16) implies $n = (a+b)k - b$ and that case (17) implies $n = (a + b)k$. For case (16) we obtain as sum:

$$\begin{aligned}
\sigma &= a\,(1 + 3 + \ldots + (2\,k - 1)) + b\,(2 + 4 + \ldots + 2\,k - 2) \\
&= a\,k^2 + b\,(k - 1)\,k \\
&= (a + b)\,k^2 - b\,k \\
&= n\,k\,,
\end{aligned}$$

while for case (17) we obtain as sum:

$$\begin{aligned}
\sigma &= a\,(1 + 3 + \ldots + (2\,k - 1)) + b\,(2 + 4 + \ldots + 2\,k) \\
&= a\,k^2 + b\,(k + 1)\,k \\
&= (a + b)\,k^2 + b\,k \\
&= (n + b)\,k\,.
\end{aligned}$$

The above proves that if π_1 is an optimal strategy, one of the five conditions from Theorem 4 is satisfied. On the other hand, it is obvious that whenever one of those five conditions is satisfied, there exists an optimal strategy, which concludes the proof of Theorem 4. The above reasoning also proves the statements in Propositions 6 and 8 about the tally-representation of the optimal strategies.

We still need to prove that conditions (6) and (7) are mutually exclusive. Therefore, suppose there exist a, b, k satisfying (6) and a', b', k' satisfying (7) (adding accents where appropriate). We then have that $(n + b')k' = \sigma = nk$, which implies $b'k' = n(k - k')$. As $n = (a' + b')k'$, it follows that $n = a'k' +$

$n(k-k')$ and either $k=k'$ which implies $b'=0$, or $k=k'+1$ which implies $a'=0$. But $a'=0$ and $b'=0$ were excluded in (7) and both conditions are therefore mutually exclusive.

4.6 Finalizing the proof of Proposition 6

To completely prove Proposition 6, we must still determine the number of optimal strategies in a game of type (6).

Assume a, b, k are solutions of (6), k obviously being invariant for all solutions as this follows immediately from (6). Suppose now that $n=(a+b)\,k-b$ and $n=(a'+b')\,k-b'$. It follows that $(a'-a)\,k=(b-b')\,(k-1)$. Since these are all integers, this is equivalent to $a'=a+l\,(k-1)$ and $b'=b-l\,k$, for some integer l. Restricting a' and b' such that $a'\geq 0$ and $b'\geq 0$ we obtain that l can vary from $-\lfloor a/(k-1)\rfloor$ to $\lfloor b/k\rfloor$ and there are indeed exactly $\lfloor a/(k-1)\rfloor+\lfloor b/k\rfloor+1$ solutions.

4.7 Finalizing the proof of Proposition 8

Finally, the proof of Proposition 8 is concluded by showing that there is exactly one optimal strategy in a game of type (7). There exist a, b, k for which $n=(a+b)\,k$, $\sigma=(n+b)\,k$, $a\neq 0$ and $b\neq 0$. Suppose $\pi_1=(i_1,\ldots,i_n)$ and $\pi_2=(i'_1,\ldots,i'_n)$ are two different optimal strategies. As was shown above, both partitions must be of type (17), with a and b different from 0.

Suppose $i_n<i'_n$. Using increment/decrement operations we can construct π_2 from π_1, only decrementing numbers smaller than or equal to i_n, and in at least one of these intermediate steps, transforming the intermediate partition π_i to π'_i, the number i_n will be incremented and we therefore obtain $Q_{\pi_1,\pi_2}\geq Q_{\pi_1,\pi'_i}=Q_{\pi_1,\pi_i}+\frac{a}{2n^2}>Q_{\pi_1,\pi_i}\geq Q_{\pi_1,\pi_1}=1/2$, implying $Q_{\pi_1,\pi_2}>1/2$ and π_2 is then not an optimal strategy. The case $i_n>i'_n$ is completely analogous. Therefore, k is an invariant of the solution space of (17), which implies that the values of a and b are also fixed. Thus, there exists only one optimal strategy in games of type (17).

As mentioned before, it is obvious that Proposition 6 (resp. Proposition 8) implies Corollary 10 (resp. Corollary 11).

This completes the proof of the theorem, propositions and corollaries of Section 3.

5 Conclusion

We have determined the necessary and sufficient conditions for a strategy to be optimal in an (n, σ) dice game, where the strategies consist of fair dice with n faces and the number of eyes distributed over the faces, on each face at least one eye, equal to σ. A complete characterization of the games containing optimal strategies as well as the explicit form of the optimal strategies themselves has been obtained.

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