Extreme Copulas and the Comparison of Ordered Lists

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Abstract

We introduce two extreme methods to pairwisely compare ordered lists of the same length, viz. the comonotonic and the countermonotonic comparison method, and show that these methods are respectively related to the copula $T_{\mathbf{M}}$ (the minimum operator) and the Łukasiewicz copula $T_{\mathbf{L}}$ used to join marginal cumulative distribution functions into bivariate cumulative distribution functions. Given a collection of ordered lists of the same length, we generate by means of $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ two probabilistic relations $Q^{\mathbf{M}}$ and $Q^{\mathbf{L}}$ and identify their type of transitivity. Finally, it is shown that any probabilistic relation with rational elements on a 3-dimensional space of alternatives which possesses one of these types of transitivity, can be generated by three ordered lists and at least one of the two extreme comparison methods.

Keywords: Comonotonic/Countermonotonic comparison, Copula, Ordered list, Probabilistic relation, Transitivity.

1 Introduction

Recently, we have introduced the notion of a dice model as a framework for describing a class of probabilistic relations [5]. Recall that a probabilistic relation Q on a set of alternatives A, often also called a reciprocal or ipsodual relation, is a mapping from A^2 to $[0,1]$ such that $Q(a,b)+Q(b,a)=1$. Probabilistic relations are frequently used in various types of preference models [2, 7, 12]. The number $Q(a,b)$ can, for instance, express the degree of preference of alternative a over alternative b.

Central to the dice model is a collection $(X_1, X_2, \ldots, X_m)$ of $m > 2$ hypothetical fair dice, each dice possessing n faces, and each face containing a strictly positive integer. For any couple of dice (X_i, X_j) taken from the collection, the probability q_{ij} that X_i wins from X_j can be computed, conform with the principle that when the dice are rolled independently, the one showing the highest number wins, whereas in case of a tie the dice are rolled again in order to pinpoint a winner. The dice can therefore also be regarded as independent discrete random variables X_i uniformly distributed on multisets A_i of cardinality n, where A_i contains all the integers on the faces of dice X_i – note that a multiset can contain a same element more than once. By statistically comparing any two random variables X_i and X_j from the collection $(X_1, X_2, \ldots, X_m)$, a probabilistic relation $Q^{\mathbf{P}} = [q^{\mathbf{P}}_{ij}]$ is generated in the following way [5, 6]:

$$q^{\mathbf{P}}_{ij} = \text{Prob}\{X_i > X_j\} + \frac{1}{2}\text{Prob}\{X_i = X_j\}\,. \tag{1}$$

The meaning of the superscript $\mathbf{P}$ will be made clear later on. In agreement with the dice metaphor, the winning probabilities $q^{\mathbf{P}}_{ij}$ are computed as:

$$q^{\mathbf{P}}_{ij} = \frac{\#\{(u,v) \in A_i \times A_j \mid u > v\}}{n^2} + \frac{\#\{(u,v) \in A_i \times A_j \mid u = v\}}{2n^2}\,. \tag{2}$$

In the language of preference modeling, one should identify the set of alternatives with the collection of dice, the preferences being winning probabilities.

In [5], our main interest was to study the transitivity of the probabilistic relation $Q^{\mathbf{P}}$. In particular, the relation $Q^{\mathbf{P}}$ exhibits a type of transitivity, called dice-transitivity, which does not belong to the class of stochastic transitivity [7], neither to the class of T-transitivity, with T a t-norm [9]. Instead, it nicely fits into the framework of cycle-transitivity [4], which generalizes, like the FG-transitivity framework of Switalski [13], the concepts of stochastic transitivity and T-transitivity, but is even more general since the latter does not cover the case of dice-transitivity [3].

Later on the dice model has been extended to yield a framework for the pairwise comparison of independent discrete or continuous random variables with arbitrary marginal distributions [6]. Any collection $(X_1, X_2, \ldots, X_m)$ of independent random variables still generates a probabilistic relation $Q^{\mathbf{P}}$ through the application of the probabilistic definition (1). The main result of [6] is that this probabilistic relation is always dice-transitive, proving that the particular case of independent discrete random variables uniformly distributed on integer multisets is a generic case, as far as the transitivity of the generated probabilistic relation is concerned.

Obviously, one can invent many alternatives to pairwisely compare multisets of cardinality n. In this paper, two such alternatives are considered. By ordering the elements – without loss of generality, we systematically opt for the increasing order – the multisets are transformed into ordered lists of integers, all lists having length n. The winner of two ordered lists is determined as follows. A number $k \in \mathbb{N}[1, n]$ is randomly chosen, where $\mathbb{N}[a, b]$ denotes the set of integers in the closed interval $[a, b]$. Two extreme list comparison strategies are distinguished. With the comonotonic (countermonotonic) comparison strategy the integer at position k in the first list is compared to the integer at position k (at position $n - k + 1$) in the second list; unless these integers are equal, in which case there is a replay, the list that contains the highest integer at the respective position is the winning list.

In the statistical interpretation, ordered lists can be regarded as uniformly distributed discrete random variables, and the winning probabilities corresponding to either the comonotonic or the countermonotonic comparison strategy can be used to generate from a given collection $(X_1, X_2, \ldots, X_m)$ of such random variables a probabilistic relation, which will be respectively denoted as $Q^{\mathbf{M}}$ and $Q^{\mathbf{L}}$. It will be shown that the comonotonic and countermonotonic comparison strategies amount to treating any two random variables X_i and X_j that are compared as being fictitiously coupled, where the type of coupling, which is most elegantly described by a copula C_{ij}, depends upon the comparison strategy. The copula C_{ij} is the function that joins the one-dimensional marginal cumulative distribution functions F_{X_i} and F_{X_j} into a bivariate cumulative distribution function F_{X_i,X_j}[11]. It is one of our concerns to lay bare the relationship between the comparison strategies and particular copulas. The main results of the paper are, however, related to the characterization of the transitivity of the probabilistic relations $Q^{\mathbf{M}}$ and $Q^{\mathbf{L}}$.

The paper is organized as follows. In the next section, the two extreme strategies for comparing ordered lists of integers of the same cardinality n are introduced and it is shown how these strategies both lead to a probabilistic relation. In Section 3 the random variable interpretation of ordered lists is introduced and the relationship between the list comparison strategies and specific copulas for pairwisely coupling the random variables is analyzed. In Section 4, the definition of cycle-transitivity is recalled and emphasis is put on the way this framework generalizes both stochastic transitivity and T-transitivity. In Sections 5 and 6, which contain the main results of this paper, the transitivity of the probabilistic relations generated by both comparison strategies is analyzed and shown to fit into the cycle-transitivity framework. Finally, some conclusions and prospects for future work are presented in Section 7.

2 Two extreme list comparison strategies

As stated before, we look for methods to compare ordered lists of numbers. For simplicity, we assume that the lists all have the same length n, with n arbitrary but fixed. As the comparison of lists will finally amount to the comparison of numbers, and since the lists have finite length, we can assume without loss of generality that the numbers are (strictly positive) integers.

Let V_n denote the class of all lists of length n composed of strictly positive integers that are listed in increasing order. An ordered list $X \in V_n$ is denoted as $(x^{(1)}, x^{(2)}, \ldots, x^{(n)})$, where all $x^{(j)} \in \mathbb{N}_0$ and $x^{(1)} \leq x^{(2)} \leq \cdots \leq x^{(n)}$. To X is associated the multiset $A = \{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\}$ and (with some abuse of notation) the discrete random variable uniformly distributed on the multiset A will also be noted as X. Any $(X_1, X_2, \ldots, X_m) \in V_n^m$ is called a (ordered) collection of ordered lists, or equivalently, a collection of discrete random variables, uniformly distributed respectively on the multisets A_i, where A_i denotes the multiset associated to the ordered list X_i.

As was already mentioned in the introduction, the comonotonic comparison strategy of two ordered lists of length n consists in selecting at random a list position in the range from 1 to n and in further comparing the integers at this position in both lists. The probability that an ordered list wins from another ordered list, is again the key concept in building for any collection $(X_1, X_2, \ldots, X_m) \in V_n^m$ of ordered lists, a probabilistic relation, which is now denoted $Q^{\mathbf{M}}$.

Definition 2.1 Any collection $(X_1, X_2, \ldots, X_m) \in V_n^m$ of ordered lists generates according to the comonotonic comparison strategy of lists, a probabilistic relation $Q^{\mathbf{M}} = [q_{ij}^{\mathbf{M}}]$ that is computed as:

$$q_{ij}^{\mathbf{M}} = \frac{\#\{k \in \mathbb{N}[1,n] \mid x_i^{(k)} > x_j^{(k)}\}}{n} + \frac{\#\{k \in \mathbb{N}[1,n] \mid x_i^{(k)} = x_j^{(k)}\}}{2n} . \qquad (3)$$

Note that for two identical ordered lists, the probability that one of the two wins from the other equals 1/2, which agrees with the fact that there is no circumstance that any one of the two can ever win from the other. Of course, the reverse is not always true; two ordered lists that have equal probability 1/2 to win from each other, are not necessarily identical.

If instead of comparing an element selected at random from the first ordered list to the element at the same position in the second ordered list, one compares the randomly selected element from the first list to a randomly selected element from the second list, then the comparison strategy becomes the same as the one between two dice with the same number of faces. Note that the ordering of the lists then becomes obsolete. Hence, the comparison strategy for dice can be applied to a collection of ordered lists too and the probabilistic relation that is generated is $Q^{\mathbf{P}}$, which in the framework of the dice model is defined in (2), but with the list notations reads:

$$q_{ij}^{\mathbf{P}} = \frac{\#\{(k,l) \in \mathbb{N}[1,n]^2 \mid x_i^{(k)} > x_j^{(l)}\}}{n^2} + \frac{\#\{(k,l) \in \mathbb{N}[1,n]^2 \mid x_i^{(k)} = x_j^{(l)}\}}{2n^2}. \tag{4}$$

The list comparison strategy from which the probabilistic relation $Q^{\mathbf{P}}$ originates, will be called the independent comparison strategy.

With respect to the above strategyy, the comonotonic comparison strategy is mirrored in what is called the countermonotonic comparison strategy. Applied to ordered lists of length n, it consists in selecting at random a list position k in the range from 1 to n and in comparing the integer at position k in the first list to the integer at position $n-k+1$ in the second list. Note that if the second list would be ordered in decreasing, instead of increasing order, then the rules of the comonotonic strategy could be applied, which justifies the name countermonotonic. For any collection $(X_1, X_2, \ldots, X_m) \in V_n^m$ the countermonotonic comparison leads again to a probabilistic relation, which is denoted $Q^{\mathbf{L}}$.

Definition 2.2 Any collection $(X_1, X_2, \ldots, X_m) \in V_n^m$ of ordered lists generates according to the countermonotonic comparison strategy of lists, a probabilistic relation $Q^{\mathbf{L}} = [q_{ij}^{\mathbf{L}}]$ that is computed as:

$$q_{ij}^{\mathbf{L}} = \frac{\#\{k \in \mathbb{N}[1,n] \mid x_i^{(k)} > x_j^{(n+k-1)}\}}{n} + \frac{\#\{k \in \mathbb{N}[1,n] \mid x_i^{(k)} = x_j^{(n+k-1)}\}}{2n}. \tag{5}$$

Let us illustrate the comparison of ordered lists for the three mentioned strategies on the same example of two ordered lists X_i and X_j with associated multisets $A_i = \{1, 2, 5, 8\}$ and $A_j = \{2, 3, 5, 7\}$. As illustrated in Figure 1(a), the computation of $q_{ij}^{\mathbf{P}}$ amounts to the computation of the winning probability between X_i and X_j as if they were fair four-faced dice. We clearly obtain that $q_{ij}^{\mathbf{P}} = (0 + 0.5 + 2.5 + 4)/16 = 7/16$. In Figure 1(b), the comonotonic comparison strategy is illustrated and we obtain $q_{ij}^{\mathbf{M}} = 0+0+1/8+1/4 = 3/8$. Finally, in Figure 1(c), the countermonotonic strategy is illustrated, which clearly yields $q_{ij}^{\mathbf{L}} = 1/2$. For this particular example, it holds that $q_{ij}^{\mathbf{M}} \leq q_{ij}^{\mathbf{P}} \leq q_{ij}^{\mathbf{L}}$. This ordering of the probabilistic relations $Q^{\mathbf{M}}$, $Q^{\mathbf{P}}$ and $Q^{\mathbf{L}}$ should, however, not be taken as a rule, as it is easy to construct counterexamples.

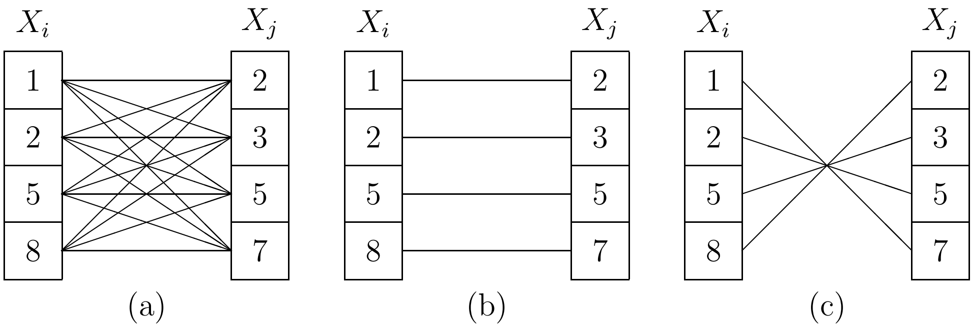

Figure 1: Various methods for comparing two ordered lists illustrated on the same example lists.

3 Connection between list comparison strategies and copulas

Let us focus on the random variable interpretation of ordered lists, in the sense that any ordered list can be regarded as a discrete random variable that is uniformly distributed on the multiset associated to the list.

It is well known that for discrete random variables X_i and X_j that are distributed on multisets of (strictly) positive integers, the probability $p_{X_i,X_j}(k,l)$ that X_i takes integer value k and X_j takes integer value l, can be obtained from the joint cumulative distribution function F_{X_i,X_j} as follows:

$$p_{X_i,X_j}(k,l) = F_{X_i,X_j}(k,l)+F_{X_i,X_j}(k-1,l-1)-F_{X_i,X_j}(k,l-1)-F_{X_i,X_j}(k-1,l)\,.$$

Sklar's theorem [11] says that if a joint cumulative distribution function F_{X_i,X_j} has marginals F_{X_i} and F_{X_j}, then there exists a copula C such that for all x, y:

$$F_{X_i,X_j}(x,y) = C_{ij}(F_{X_i}(x), F_{X_j}(y))\,. \tag{6}$$

On the other hand, if C_{ij} is a copula and F_{X_i} and F_{X_j} are cumulative distribution functions, then the function defined by (6) is a joint cumulative distribution function with marginals F_{X_i} and F_{X_j}.

Let us recall [11] that a copula is a binary operation $C : [0,1]^2 \to [0,1]$ that has neutral element 1 and absorbing element 0, and that satisfies the property of moderate growth: for any $(x_1, x_2, y_1, y_2) \in [0,1]^4$

$$(x_1 \leq x_2 \wedge y_1 \leq y_2) \Rightarrow C(x_1, y_1) + C(x_2, y_2) \geq C(x_1, y_2) + C(x_2, y_1)\,.$$

All copulas are situated between the Łukasiewicz copula $T_{\mathbf{L}}(x,y) = \max(0, x+y-1)$ and the minimum copula $T_{\mathbf{M}}(x,y) = \min(x,y)$. In the literature on copulas, these two extreme copulas are usually denoted as W and M. Here we prefer to use the notation that refers to the fact that these copulas are also t-norms. The same holds for $T_{\mathbf{P}}$, the ordinary product, which is a t-norm and a copula as well.

Let us now lay bare the connection between the probabilistic relation $Q^{\mathbf{M}}$ ($Q^{\mathbf{L}}$) and the copula $T_{\mathbf{M}}$ ($T_{\mathbf{L}}$). Whatever comparison strategy, as soon as two dice or two ordered lists are interpreted as discrete random variables X_i, X_j, the winning probability used to compare these random variables and to set up a probabilistic relation, is defined by $\text{Prob}\{X_i > X_j\} + \text{Prob}\{X_i = X_j\}/2$. For discrete random variables, this equality can be restated as

$$q_{ij} = \sum_{k>l} p_{X_i,X_j}(k,l) + \frac{1}{2}\sum_{k=l} p_{X_i,X_j}(k,l)\,. \tag{7}$$

The random variable X_i (X_j) has cumulative distribution function F_{X_i} (F_{X_j}) and probability mass function p_{X_i} (p_{X_j}). We first consider the probabilistic relation when the copula $T_{\mathbf{M}}$ is used. It then holds that $p^{\mathbf{M}}_{X_i,X_j}(0,0) = 0$ and for all $(k,l) \in \mathbb{N}_0^2$:

$$\begin{aligned} p^{\mathbf{M}}_{X_i,X_j}(k,l) &= \min(F_{X_i}(k), F_{X_j}(l)) + \min(F_{X_i}(k-1), F_{X_j}(l-1)) \\ &\quad - \min(F_{X_i}(k), F_{X_j}(l-1)) - \min(F_{X_i}(k-1), F_{X_j}(l))\,, \end{aligned}$$

which is equivalent to:

$$p^{\mathbf{M}}_{X_i,X_j}(k,l) = \begin{cases} 0 & \text{, if } F_{X_i}(k) \leq F_{X_j}(l-1) \vee F_{X_j}(l) \leq F_{X_i}(k-1)\,, \\ \min(F_{X_i}(k), F_{X_j}(l)) - \max(F_{X_i}(k-1), F_{X_j}(l-1)) & \\ & \text{, otherwise}\,. \end{cases}$$

As each element in the multiset has probability $1/n$, the first line of the above expression is equivalent to saying that when $\#\{t \in \mathbb{N}[1,n] \mid x_i^{(t)} = k \wedge x_j^{(t)} = l\} = 0$, it holds that $p^{\mathbf{M}}_{X_i,X_j}(k,l) = 0$. The second line is then equivalent to saying that when $\#\{t \in \mathbb{N}[1,n] \mid x_i^{(t)} = k \wedge x_j^{(t)} = l\} = u > 0$, it holds that $p^{\mathbf{M}}_{X_i,X_j}(k,l) = u/n$. Using (7), expression (3) for $Q^{\mathbf{M}}$ now follows immediately.

Secondly, considering the copula $T_{\mathbf{L}}$, we obtain $p^{\mathbf{L}}_{X_i,X_j}(0,0) = 0$ and for all $(k,l) \in \mathbb{N}_0^2$:

$$p^{\mathbf{L}}_{X_i,X_j}(k,l) =$$

$$\max(0, F_{X_i}(k) + F_{X_j}(l) - 1) + \max(0, F_{X_i}(k-1) + F_{X_j}(l-1) - 1)$$
$$- \max(0, F_{X_i}(k) + F_{X_j}(l-1) - 1) - \max(0, F_{X_i}(k-1) + F_{X_j}(l) - 1)\,,$$

which is equivalent to:

$$p^{\mathbf{L}}_{X_i,X_j}(k,l) = \begin{cases} 0 \ , \ \text{if } F_{X_i}(k) \leq 1 - F_{X_j}(l) \vee 1 - F_{X_j}(l-1) \leq F_{X_i}(k-1)\,, \\ \min(F_{X_i}(k), 1 - F_{X_j}(l-1)) - \max(F_{X_i}(k-1), 1 - F_{X_j}(l)) \\ \quad , \ \text{otherwise}\,. \end{cases}$$

The first line of the above expression is equivalent to demanding that when $\#\{t \in \mathbb{N}[1,n] \mid x_i^{(t)} = k \wedge x_j^{(n+t-1)} = l\} = 0$, it holds that $p^{\mathbf{L}}_{X_i,X_j}(k,l) = 0$. The second part is then equivalent to saying that when $\#\{t \in \mathbb{N}[1,n] \mid x_i^{(t)} = k \wedge x_j^{(n+t-1)} = l\} = u > 0$, it holds that $p^{\mathbf{L}}_{X_i,X_j}(k,l) = u/n$. Using (7), expression (5) for $Q^{\mathbf{L}}$ follows immediately.

Finally, if we choose the ordinary product $T^{\mathbf{P}}$ as copula, then the bivariate cumulative distribution function is the product of two one-dimensional marginal cumulative distribution functions. This is equivalent to stating that the random variables are independent. Hence, for all $(k,l) \in \mathbb{N}_0^2$ it holds that:

$$p^{\mathbf{P}}_{X_i,X_j}(k,l) = p_{X_i}(k)\, p_{X_j}(l)\,.$$

Substitution in (7) yields expression (4) for $Q^{\mathbf{P}}$.

We can conclude that there is a one-to-one relationship between the independent, the comonotonic and the countermonotonic comparison of ordered lists on the one side and the copulas $T_{\mathbf{P}}, T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ on the other side. This relationship not only justifies the superscripts in the probabilistic relations $Q^{\mathbf{P}}, Q^{\mathbf{M}}$ and $Q^{\mathbf{L}}$, it also justifies the name of the comparison strategies. Indeed, if one samples a couple of random variables (X,Y) whose distribution functions are joined into a bivariate distribution function by means of $T_{\mathbf{M}}$, then for any two sample values (x_1,y_1) and (x_2,y_2) it holds that $x_1 < x_2$ implies $y_1 \leq y_2$ and $x_1 > x_2$ implies $y_1 \geq y_2$. Such random variables are called comonotonic. Similarly, if one samples a couple of random variables (X,Y) whose distribution functions are joined into a bivariate distribution function by means of $T_{\mathbf{L}}$, then for any two sample values (x_1,y_1) and (x_2,y_2) it holds that $x_1 < x_2$ implies $y_1 \geq y_2$ and $x_1 > x_2$ implies $y_1 \leq y_2$; such random variables are called countermonotonic.

4 Cycle-transitivity

For a probabilistic relation Q, we define for all $(a,b,c) \in A^3$ the following quantities [4]:

$$\alpha_{abc} = \min(Q(a,b), Q(b,c), Q(c,a)) \,,$$
$$\beta_{abc} = \text{median}(Q(a,b), Q(b,c), Q(c,a)) \,,$$
$$\gamma_{abc} = \max(Q(a,b), Q(b,c), Q(c,a)) \,.$$

Let us also denote $\Delta = \{(x,y,z) \in [0,1]^3 \mid x \le y \le z\}$.

Definition 4.1 A function $U : \Delta \to \mathbb{R}$ is called an upper bound function if it satisfies:

(i) $U(0,0,1) \ge 0$ and $U(0,1,1) \ge 1$;

(ii) for any $(\alpha, \beta, \gamma) \in \Delta$:

$$U(\alpha,\beta,\gamma) + U(1-\gamma, 1-\beta, 1-\alpha) \ge 1 \,. \tag{8}$$

The function $L : \Delta \to \mathbb{R}$ defined by

$$L(\alpha,\beta,\gamma) = 1 - U(1-\gamma, 1-\beta, 1-\alpha)$$

is called the dual lower bound function of a given upper bound function U. Inequality (8) then simply expresses that $L \le U$.

Definition 4.2 A probabilistic relation Q on A is called cycle-transitive w.r.t. an upper bound function U if for any $(a,b,c) \in A^3$ it holds that

$$L(\alpha_{abc}, \beta_{abc}, \gamma_{abc}) \le \alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \le U(\alpha_{abc}, \beta_{abc}, \gamma_{abc}) \,, \tag{9}$$

where L is the dual lower bound function of U.

In practice, it is sufficient to check (9) for a single permutation of any $(a,b,c) \in A^3$. Alternatively, it is also sufficient to verify the right-hand inequality (or equivalently, the left-hand inequality) for two permutations of any $(a,b,c) \in A^3$ (not being cyclic permutations of one another), e.g. (a,b,c) and (c,b,a). Hence, (9) can be replaced by

$$\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \le U(\alpha_{abc}, \beta_{abc}, \gamma_{abc}) \,. \tag{10}$$

Note that a value of $U(\alpha,\beta,\gamma)$ equal to 2 is used to express that for the given values there is no restriction at all.

For two upper bound functions such that $U_1 \leq U_2$, it clearly holds that cycle-transitivity w.r.t. U_1 implies cycle-transitivity w.r.t. U_2. It is clear that $U_1 \leq U_2$ is not a necessary condition for the latter implication to hold. Two upper bound functions U_1 and U_2 will be called *equivalent* if for any $(\alpha, \beta, \gamma) \in \Delta$ it holds that

$$\alpha + \beta + \gamma - 1 \leq U_1(\alpha, \beta, \gamma)$$

is equivalent to

$$\alpha + \beta + \gamma - 1 \leq U_2(\alpha, \beta, \gamma)\,.$$

Cycle-transitivity includes as special cases T-transitivity and all known types of g-stochastic transitivity. A $[0,1]$-valued relation R on a set of alternatives A is called T-transitive [8] if for any $(a, b, c) \in A^3$ it holds that $T(R(a, b), R(b, c)) \leq R(a, c)$. The following proposition [4] shows how T-transitivity fits into the framework of cycle-transitivity in case the t-norm T is 1-Lipschitz continuous (for short, 1-Lipschitz), which means that for all $(x, y, z) \in [0, 1]^3$ it holds that $|T(x, y) - T(x, z)| \leq |y - z|$.

Proposition 4.1 *Let T be a 1-Lipschitz t-norm. A probabilistic relation is T-transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_T defined by*

$$U_T(\alpha, \beta, \gamma) = \alpha + \beta - T(\alpha, \beta)\,. \tag{11}$$

Note that 1-Lipschitz t-norms can also be regarded as associative and commutative copulas. The special t-norms $T_{\mathbf{M}}, T_{\mathbf{P}}$ and $T_{\mathbf{L}}$ are examples of 1-Lipschitz t-norms. By means of (11) we immediately find that $T_{\mathbf{M}}$-transitivity, $T_{\mathbf{P}}$-transitivity and $T_{\mathbf{L}}$-transitivity are equivalent to cycle-transitivity w.r.t. the upper bound functions $U_{\mathbf{M}}(\alpha, \beta, \gamma) = \beta$, $U_{\mathbf{P}}(\alpha, \beta, \gamma) = \alpha + \beta - \alpha\beta$ and $U_{\mathbf{L}}(\alpha, \beta, \gamma) = \min(\alpha + \beta, 1)$, respectively. For the case of $T_{\mathbf{L}}$-transitivity, an equivalent upper bound function is given by $U'_{\mathbf{L}}(\alpha, \beta, \gamma) = 1$.

In the literature one finds various types of stochastic transitivity [2, 10]. They can, however, be regarded as special cases of a generic type of stochastic transitivity, which we have called g-stochastic transitivity. Let g be a commutative, increasing $[1/2, 1]^2 \to [1/2, 1]$ mapping. A probabilistic relation Q on A is called g-stochastic transitive if for any $(a, b, c) \in A^3$ it holds that

$$(Q(a, b) \geq 1/2 \ \wedge \ Q(b, c) \geq 1/2) \ \Rightarrow \ Q(a, c) \geq g(Q(a, b), Q(b, c))\,.$$

In [4], we have proven the following proposition.

Proposition 4.2 *Let g be a commutative, increasing $[1/2, 1]^2 \to [1/2, 1]$ mapping such that $g(1/2, x) \leq x$ for any $x \in [1/2, 1]$. A probabilistic relation* *is g-stochastic transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_g defined by*

$$U_g(\alpha,\beta,\gamma) = \begin{cases} \beta+\gamma-g(\beta,\gamma) & \text{, if } \beta \geq 1/2 \wedge \alpha < 1/2\,, \\ 1/2 & \text{, if } \alpha \geq 1/2\,, \\ 2 & \text{, if } \beta < 1/2\,. \end{cases} \tag{12}$$

We obtain as special cases (only mentioning the function g):

(i) strong stochastic transitivity: $g_{ss}(\beta,\gamma) = \max(\beta,\gamma) = \gamma$;

(ii) moderate stochastic transitivity: $g_{ms}(\beta,\gamma) = \min(\beta,\gamma) = \beta$;

(iii) weak stochastic transitivity: $g_{ws}(\beta,\gamma) = 1/2$.

The probabilistic relation $Q^{\mathbf{P}}$ generated by a dice model has been shown to be dice-transitive [5, 6]. This is a special type of cycle-transitivity which is neither a type of T-transitivity nor a type of g-stochastic transitivity.

Definition 4.3 A probabilistic relation is dice-transitive if it is cycle-transitive w.r.t. the upper bound function U_D defined by

$$U_D(\alpha,\beta,\gamma) = \beta+\gamma-\beta\gamma\,. \tag{13}$$

Note that dice-transitivity can be situated between $T_{\mathbf{P}}$-transitivity and $T_{\mathbf{L}}$-transitivity, and also between moderate stochastic transitivity and $T_{\mathbf{L}}$-transitivity.

5 Transitivity of $Q^{\mathbf{M}}$

In this section our aim is to characterize the type of transitivity of the probabilistic relation $Q^{\mathbf{M}}$ generated from a collection of ordered lists by applying the comonotonic list comparison strategy.

Proposition 5.1 *The probabilistic relation $Q^{\mathbf{M}}$ generated from the pairwise comonotonic comparison of a collection of ordered lists, is $T_{\mathbf{L}}$-transitive.*

Proof. Consider any three ordered lists $X_i = (x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(n)})$, $X_j = (x_j^{(1)}, x_j^{(2)}, \ldots, x_j^{(n)})$, and $X_k = (x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(n)})$, taken from a given collection $(X_1, X_2, \ldots, X_m)$ of ordered lists. From (3) it follows that we only need to compare elements of the same rank in the lists, i.e. the triplets $(x_i^{(l)}, x_j^{(l)}, x_k^{(l)})$, for all $l \in \mathbb{N}[1,n]$. It is obvious that the specific comparison done for each such triplet contributes at least $1/n$ and at most $2/n$

to the sum $q_{ij}^{\mathbf{M}} + q_{jk}^{\mathbf{M}} + q_{ki}^{\mathbf{M}}$. Summing over the n positions, we obtain $1 \leq q_{ij}^{\mathbf{M}} + q_{jk}^{\mathbf{M}} + q_{ki}^{\mathbf{M}} \leq 2$. This proves that Q is cycle-transitive w.r.t. the upper bound function $U(\alpha, \beta, \gamma) = 1$, whence $Q^{\mathbf{M}}$ is $T_{\mathbf{L}}$-transitive. □

Note that we do not really need the cycle-transitivity framework to cover the type of transitivity exhibited by $Q^{\mathbf{M}}$. Nonetheless, for situating the $T_{\mathbf{L}}$-transitivity of $Q^{\mathbf{M}}$ with respect to dice-transitivity (the type of transitivity of $Q^{\mathbf{P}}$) and the type of transitivity of $Q^{\mathbf{L}}$ which we will characterize further on, it is preferable to cast all these types of transitivity into their equivalent cycle-transitive form.

In the case of the dice model, we have been able to formulate conditions under which a dice-transitive probabilistic relation can be generated by a dice model. By analogy, we attempt to find out whether any $T_{\mathbf{L}}$-transitive probabilistic relation can be generated by a collection of ordered lists when applying the comonotonic comparison strategy. The following two propositions complete this analysis.

Proposition 5.2 *Any 3-dimensional $T_{\mathbf{L}}$-transitive probabilistic relation $Q = [q_{ij}]$ with rational elements can be generated by the application of the comonotonic comparison strategy to a collection of three ordered lists of the same length and such that the associated multisets are disjoint.*

Proof. Let Q be such that $q_{12} = p/n$, $q_{23} = q/n$, $q_{31} = r/n$. We will construct three ordered lists $X_1 = (x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(n)})$, $X_2 = (x_2^{(1)}, x_2^{(2)}, \ldots, x_2^{(n)})$, and $X_3 = (x_3^{(1)}, x_3^{(2)}, \ldots, x_3^{(n)})$ of length n, all with different elements and such that the associated sets A_1, A_2 and A_3 are disjoint. More specifically, we take $\{x_1^{(j)}, x_2^{(j)}, x_3^{(j)}\} = \{3j-2, 3j-1, 3j\}$ for all $j \in \mathbb{N}[1, n]$. Furthermore, choose $x_1^{(j)} > x_2^{(j)}$ for $j \in \mathbb{N}[1, p]$, $x_1^{(j)} < x_2^{(j)}$ for $j \in \mathbb{N}[p+1, n]$, $x_2^{(j)} < x_3^{(j)}$ for $j \in \mathbb{N}[1, n-q]$ and $x_2^{(j)} > x_3^{(j)}$ for $j \in \mathbb{N}[n-q+1, n]$. It therefore already holds that $x_3^{(j)} > x_1^{(j)}$ for $j \in \mathbb{N}[p+1, n-q]$ and $x_3^{(j)} < x_1^{(j)}$ for $j \in \mathbb{N}[n-q+1, p]$. However, for $j \notin \mathbb{N}[p+1, n-q] \cup \mathbb{N}[n-q+1, p]$, it can be chosen freely whether $x_3^{(j)} > x_1^{(j)}$ or $x_3^{(j)} < x_1^{(j)}$. As Q is $T_{\mathbf{L}}$-transitive it holds that $n-p-q \leq r \leq 2n-p-q$, and we can therefore choose enough of these remaining j such that $x_3^{(j)} > x_1^{(j)}$ holds and such that $q_{31} = r/n$, concluding the proof. □

The question arises whether the reverse property which holds for 3-dimensional $T_{\mathbf{L}}$-transitive probabilistic relations, extends to higher-dimensional relations. The question must be answered negatively, as has been pointed out by Switalski [13] in his analysis of the type of transitivity of the so-called multidimensional model.

Definition 5.1 The multidimensional model is a preference model in which the preferences generate a probabilistic relation $Q = [q_{ij}]$ defined by

$$q_{ij} = \sum_{t=1}^{n} \mu_t \, q_{ij}^{(t)} \,, \tag{14}$$

with $q_{ij}^{(t)} \in \{0, 1/2, 1\}$, $q_{ij}^{(t)} = 1 - q_{ji}^{(t)}$ for all $t \in \mathbb{N}[1, n]$ and where the weights μ_t (associated to criterion t) are such that $\mu_t \geq 0$ for all $t \in \mathbb{N}[1, n]$ and $\sum_{t=1}^{n} \mu_t = 1$.

One easily sees that a multidimensional model with all $\mu_i = 1/n$ is equivalent to a collection of ordered lists from which the probabilistic relation is generated by applying the comonotonic comparison strategy. Hence, if for a $T_{\mathbf{L}}$-transitive probabilistic relation Q with rational elements a multidimensional model can be constructed that generates Q, then this model obviously has rational weights μ_t, so that also a collection of ordered lists can be constructed that generates that same probabilistic relation Q by applying the comonotonic comparison strategy. Moreover, if no multidimensional model can be found to generate Q, then also no collection of ordered lists that generates Q exists. This problem has been shown to be closely connected to the coordinate values of the vertices of generalized transitive tournament polytopes [1]. For a review of recent results in this field, the reader is referred to [14].

From Theorem 5.3 of [13], we obtain:

Proposition 5.3 *Let $Q = [q_{ij}]$ be an m-dimensional probabilistic relation with rational elements and $m \leq 5$. Then Q is generated by a collection of ordered lists that are comonotonically compared if and only if Q is $T_{\mathbf{L}}$-transitive.*

Furthermore, it follows from [1] that for every $n > 5$ there exists a $T_{\mathbf{L}}$-transitive probabilistic relation Q that has no representation as in (14), and can therefore not be generated by ordered lists that are comonotonically compared.

6 Transitivity of $Q^{\mathbf{L}}$

We now turn to the other extreme strategy for comparing ordered lists and analyse the type of transitivity of a probabilistic relation of the type $Q^{\mathbf{L}}$. To make this analysis more transparant, we recall the notion of partial stochastic transitivity [7].

Definition 6.1 [3] A probabilistic relation Q on A is called partially stochastic transitive if for any $(a,b,c) \in A^3$ it holds that

$$(Q(a,b) > 1/2 \ \wedge \ Q(b,c) > 1/2) \ \Rightarrow \ Q(a,c) \geq \min(Q(a,b), Q(b,c))\,. \quad (15)$$

This type of transitivity resembles moderate stochastic transitivity, but is essentially different from it since moderate stochastic transitivity requires that $Q(a,c)$ in (15) is bounded from below by $\min(Q(a,b), Q(b,c))$, also when $Q(a,b) = 1/2$ or $Q(b,c) = 1/2$. Next, we recall how partial stochastic transitivity fits into the framework of cycle-transitivity [3].

Proposition 6.1 *A probabilistic relation Q is partially stochastic transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_{ps} defined by*

$$U_{ps}(\alpha,\beta,\gamma) = \gamma\,. \quad (16)$$

Note that partial stochastic transitivity is stronger than $T_{\mathbf{L}}$-transitivity, even stronger than dice-transitivity, but slightly weaker than moderate stochastic transitivity.

Proposition 6.2 *The probabilistic relation $Q^{\mathbf{L}}$ generated from the pairwise countermonotonic comparison of a collection of ordered lists, is partially stochastic transitive.*

Proof. Consider any three ordered lists $X_i = (x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(n)})$, $X_j = (x_j^{(1)}, x_j^{(2)}, \ldots, x_j^{(n)})$, and $X_k = (x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(n)})$, taken from a given collection $(X_1, X_2, \ldots, X_m)$ of ordered lists. Let us assume that $q_{ij}^{\mathbf{L}} > 1/2$ and $q_{jk}^{\mathbf{L}} > 1/2$. If $q_{ik}^{\mathbf{L}} = 1$, then $q_{ik}^{\mathbf{L}} \geq \min(q_{ij}^{\mathbf{L}}, q_{jk}^{\mathbf{L}})$ holds. Suppose now that $q_{ik}^{\mathbf{L}} < 1$ and let $u = \min\{l \mid x_i^{(l)} \geq x_j^{(n+1-l)}\}$, $v = \min\{l \mid x_j^{(l)} \geq x_k^{(n+1-l)}\}$ and $w = \min\{l \mid x_k^{(l)} \geq x_i^{(n+1-l)}\}$. If $w > \min(n-u+1, n-v+1)$, then it holds that $q_{ki}^{\mathbf{L}} \leq (n - \min(n-u+1, n-v+1))/n$, which implies that $q_{ik}^{\mathbf{L}} \geq \min(q_{ij}^{\mathbf{L}}, q_{jk}^{\mathbf{L}})$. From here on, we can therefore assume that $w \leq \min(n-u+1, n-v+1)$. As $\max(u,v) \leq n/2$, it holds that

$$x_i^{(u)} \geq x_j^{(n-u+1)} \geq x_j^{(v)} \geq x_k^{(n-v+1)} \geq x_k^{(w)} \geq x_i^{(n-w+1)} \geq x_i^{(u)}\,,$$

which implies

$$x_i^{(u)} = x_j^{(n-u+1)} = x_j^{(v)} = x_k^{(n-v+1)} = x_k^{(w)} = x_i^{(n-w+1)} = x_i^{(u)}\,.$$

We therefore have that $x_j^{(l)} = x_j^{(v)}$ for all $v \leq l \leq n-u+1$. First, suppose that $u \geq v$. For $l \in \mathbb{N}[u, n-u]$ it holds that

$$x_i^{(l)} \geq x_j^{(n-l+1)} = x_j^{(l)} = x_j^{(v)} = x_k^{(n-v+1)} \geq x_k^{(n-l+1)}\,,$$

and for $l > n - u$ it holds that (as $q^{\mathbf{L}}_{ij} > 1/2$)

$$x_i^{(l)} > x_j^{(v)} = x_k^{(n-v+1)} \geq x_k^{(n-l+1)} .$$

It follows that $q^{\mathbf{L}}_{ik} \geq q^{\mathbf{L}}_{ij}$. Finally, suppose $u < v$. For $l \in \mathbb{N}[v, n-v]$, it holds that

$$x_i^{(l)} \geq x_j^{(n-l+1)} = x_j^{(l)} \geq x_k^{(n-l+1)} ,$$

while for $l > n - v$ it holds that (as $q^{\mathbf{L}}_{jk} > 1/2$)

$$x_i^{(l)} \geq x_i^{(v)} \geq x_j^{(n-v+1)} > x_k^{(v)} \geq x_k^{(n-l+1)} .$$

It now follows that $q^{\mathbf{L}}_{ik} \geq q^{\mathbf{L}}_{jk}$. Partial stochastic transitivity must therefore be satisfied. □

For 3-dimensional partially stochastic transitive probabilistic relations the statement of Proposition 6.2 can be inverted.

Proposition 6.3 *Any 3-dimensional partially stochastic transitive probabilistic relation $Q = [q_{ij}]$ with rational elements can be generated by the application of the countermonotonic comparison strategy to a collection of three ordered lists of the same length and such that the associated multisets are disjoint.*

Proof. Let $Q = [q_{ij}]$ be such a probabilistic relation. Partial stochastic transitivity does not impose any condition only when at least two elements from $\{q_{ij}, q_{jk}, q_{ki}\}$ equal $1/2$. We first consider this case. Without loss of generality, suppose $q_{ij} = q_{jk} = 1/2$ and $q_{ki} = a/(2n)$, with $a \in \mathbb{N}[0, n]$. The following three ordered lists generate this probabilistic relation when the countermonotonic comparison strategy is applied:

$$\begin{aligned} X_i &= (\mathbb{N}[n+1, n+a], \mathbb{N}[3n+1, 5n-a]) , \\ X_j &= (\mathbb{N}[n+a+1, 2n+a], \mathbb{N}[5n-a+1, 6n-a]) , \\ X_k &= (\mathbb{N}[1, n], \mathbb{N}[2n+a+1, 3n], \mathbb{N}[6n-a+1, 6n]) . \end{aligned}$$

Suppose now that no such two elements equal $1/2$. Without loss of generality we can also assume that $\beta_{ijk} > 1/2$. As the probabilistic relation has rational elements, we can write them with common denominator n. Suppose first that the elements can be reordered such that $q_{ij} = c/n$, $q_{jk} = b/n$ and $q_{ki} = a/n$, with $c \geq b > n/2$ and $n - a \geq b$. The following three ordered lists generate this probabilistic relation:

$$\begin{aligned} X_i &= (\mathbb{N}[n+1+c-a, 2n+c-a]) , \\ X_j &= (\mathbb{N}[1, n-b], \mathbb{N}[n+1, b+c], \mathbb{N}[2n+c-a+1, 3n-a]) , \\ X_k &= (\mathbb{N}[n-b+1, n], \mathbb{N}[b+c+1, n+c-a], \mathbb{N}[3n-a+1, 3n]) . \end{aligned}$$

Secondly, suppose that the elements can be reordered such that $q_{ij} = b/n$, $q_{jk} = c/n$ and $q_{ki} = a/n$, again with $c \geq b > n/2$ and $n - a \geq b$, then this probabilistic relation is generated by the following three ordered lists:

$$\begin{aligned} X_i &= (\mathbb{N}[2n-c-a+1, 3n-c-b-a], \mathbb{N}[2n+1, 2n+b])\,, \\ X_j &= (\mathbb{N}[1, n-c], \mathbb{N}[3n-c-b+1, 2n], \mathbb{N}[2n+b+1, 3n])\,, \\ X_k &= (\mathbb{N}[n-c+1, 2n-c-a], \mathbb{N}[3n-c-b-a+1, 3n-c-b])\,. \end{aligned}$$

As all cases have been considered, the proof is complete. □

Again, the question arises whether this inverse statement can be generalized to higher-dimensional probabilistic relations. And again, the question must be answered in negative sense.

Proposition 6.4 *Not all 4-dimensional partially stochastic transitive probabilistic relations (with rational elements) can be generated from a collection of four ordered lists when the countermonotonic comparison strategy is applied.*

Proof. We will construct a class of graphs each of which represents a partially stochastic probabilistic relation that cannot be generated by a quadruplet (X_1, X_2, X_3, X_4) of ordered lists. We will use the graph of Figure 2, which shows explicitly that $q_{13} = e = 0$ and $q_{24} = f = 0$. Obviously, it holds that a, b, c, $d \in [0, 1]$.

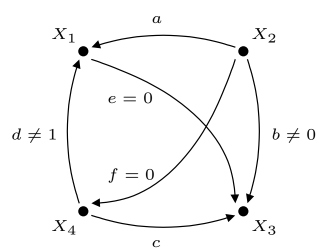

Figure 2: Partially stochastic probabilistic relations that cannot be generated by a quadruplet of ordered lists compared countermonotonically.

In this graph there are four subgraphs with three nodes. Partial stochastic transitivity has to hold for each subgraph. This leads to the following four conditions:

$$\begin{cases} 0 \leq d-a \leq \max(d, 1-a)\ , & \text{for triplet } (X_1, X_2, X_4) \\ 0 \leq d-c \leq \max(d, 1-c)\ , & \text{for triplet } (X_1, X_3, X_4) \\ 0 \leq c-b \leq \max(c, 1-b)\ , & \text{for triplet } (X_2, X_4, X_3) \\ 0 \leq a-b \leq \max(a, 1-b)\ , & \text{for triplet } (X_2, X_1, X_3) \end{cases}$$

which is equivalent to

$$\begin{cases} b \le c \le d\,, \\ b \le a \le d\,. \end{cases} \tag{17}$$

Note that these conditions can easily be satisfied. We now prove that when $e = 0$, $f = 0$, conditions (17) are fulfilled and

$$b \neq 0,\ d \neq 1\,, \tag{18}$$

we obtain a partially stochastic transitive probabilistic relation that cannot be generated by a quadruplet of ordered lists.

Let us assume that there exists such a quadruplet (X_1, X_2, X_3, X_4) of ordered lists of length n. Let $a_1 = x_1^{(n)}$and $a_2 = x_2^{(n)}$ denote the largest integer in the ordered lists X_1 and X_2, respectively. Two cases must be distinguished. In the first case we have $a_1 \geq a_2$. Since $e = 0$, the smallest integer in X_3 is strictly greater than a_1 and therefore also strictly greater than a_2 from which it follows that $b = 0$. In the second case we have $a_1 < a_2$, from which, since $f = 0$, it follows that $d = 1$. These two cases represent all possible situations and (18) does not hold in either case. Therefore, there exist no quadruplets that correspond to the partially stochastic transitive graph having the following properties:

$$b \le c \le d\,, \quad b \le a \le d\,, \quad b \neq 0\,, \quad d \neq 1\,, \quad e = 0\,, \quad f = 0\,. \tag{19}$$

Again, conditions (19) can easily be satisfied. □

7 Conclusions

We have studied two extreme ways of comparing ordered lists and established a common framework for generating from a collection of such lists a probabilistic relation. The dice model has been recognized as a third alternative for comparing ordered lists, namely the independent comparison method. A one-to-one relationship between the list comparison methods and the copulas that serve to join marginal distributions into a bivariate distribution has been laid bare. The comparison of the transitivity properties of the probabilistic relations $Q^{\mathbf{M}}$, $Q^{\mathbf{L}}$ and $Q^{\mathbf{P}}$, respectively associated to the copulas $T_{\mathbf{M}}, T_{\mathbf{L}}$ and $T_{\mathbf{P}}$, proved that $Q^{\mathbf{L}}$ has the strongest type of transitivity (partial stochastic transitivity), and that $Q^{\mathbf{M}}$ has the weakest type of transitivity ($T_{\mathbf{L}}$-transitivity).

It is our intention to further investigate the relationship between arbitrary copulas and list comparison strategies and to study the transitivity of the generated probabilistic relation. Also, we want to show that the list comparison strategies can be generalized for the sake of pairwisely comparing discrete or continuous random variables with arbitrary distributions. With regard to the transitivity of the generated probabilistic relation, it is expected that the relations $Q^{\mathbf{M}}$ and $Q^{\mathbf{L}}$, which resulted from the comparison of ordered lists, or equivalently, uniformly distributed discrete random variables, are representative for the generalized situation as well.

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