Cycle-Transitive Comparison of Independent Random Variables

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Abstract

The discrete dice model, previously introduced by the present authors, essentially amounts to the pairwise comparison of a collection of independent discrete random variables that are uniformly distributed on finite integer multisets. This pairwise comparison results in a probabilistic relation that exhibits a particular type of transitivity, called dice-transitivity. In this paper, the discrete dice model is generalized with the purpose of pairwisely comparing independent discrete or continuous random variables with arbitrary probability distributions. It is shown that the probabilistic relation generated by a collection of arbitrary independent random variables is still dice-transitive. Interestingly, this probabilistic relation can be seen as a graded alternative to the concept of stochastic dominance. Furthermore, when the marginal distributions of the random variables belong to the same parametric family of distributions, the probabilistic relation exhibits interesting types of isostochastic transitivity, such as multiplicative transitivity. Finally, the probabilistic relation generated by a collection of independent normal random variables is proven to be moderately stochastic transitive.

Key words: Comparison of independent random variables, Cycle-transitivity, Dice model, Isostochastic transitivity, Probabilistic relation, Stochastic dominance, T-norm.

1 Introduction

In the discrete dice model, recently introduced and investigated by the present authors [3], the name dice is reserved to denote a finite multiset of integers, each face of the dice having equal likelihood of showing up when the corresponding hypothetical material dice is randomly thrown. Furthermore, two dice are compared by considering the winning probability of one dice w.r.t. the other. More precisely, for any two dice A and B, we define

$$P(A,B) = \text{Prob}\{A \text{ wins from } B\} = \frac{\#\{(a,b) \in A \times B \mid a > b\}}{\#A\,\#B}\,,$$

and

$$I(A,B) = \text{Prob}\{A \text{ and } B \text{ end in a tie}\} = \frac{\#\{(a,b) \in A \times B \mid a = b\}}{\#A\,\#B}\,.$$

It then holds that

$$D(A,B) + D(B,A) = 1\,,$$

with

$$D(A,B) = P(A,B) + \frac{1}{2}I(A,B)\,.$$

We say that a relation $Q = [q_{ij}]$ is generated by a collection $(A_1, A_2, \ldots, A_m)$ of m dice, if it holds that $q_{ij} = D(A_i, A_j)$ for all (i,j). The relation Q is a probabilistic relation, also called reciprocal or ipsodual relation, and a collection of dice, together with the probabilistic relation Q it generates, is called a discrete dice model [3]. Let us recall that a probabilistic relation Q on a set of alternatives A is a mapping from A^2 to $[0,1]$, such that for all $a, b \in A$ it holds that $Q(a,b) + Q(b,a) = 1$. In general, probabilistic relations are not only a convenient tool for expressing the result of the pairwise comparison of a set of alternatives [1], but they also appear in various fields such as game theory [5], voting theory [7,12] and psychological studies on preference and discrimination in (individual or collective) decision making methods [4].

For two dice A and B, it can be stated that $A >_s B$ (A statistically wins from B) if $D(A,B) > 1/2$, and $A =_s B$ (A is statistically indifferent to B) if $D(A,B) = 1/2$. One of the main features of a discrete dice model is that its probabilistic relation can show cyclic behaviour, which means that there exist A, B, C such that $A >_s B$, $B >_s C$ and $C >_s A$.

As an example, consider the three dice A_1, A_2 and A_3 which, instead of the usual numbers, carry the following integers on their faces:

$$A_1 = \{1,3,4,15,16,17\},\ A_2 = \{2,10,11,12,13,14\},\ A_3 = \{5,6,7,8,9,18\}\,.$$

Clearly, $q_{12} = 20/36$, $q_{23} = 25/36$ and $q_{31} = 21/36$, whence $A_1 >_s A_2$, $A_2 >_s A_3$ and $A_3 >_s A_1$.

The occurence of cycles has been observed in various psychological experiments related to gambling [16], to judgement of relative pitch in music [13] and to human preferences [14], for instance. Clearly, the possible occurence of cycles implies that the relation $>_s$ derived from the probabilistic relation Q of a dice model is in general not transitive.

There is yet another way of looking at the dice, namely as discrete random variables that are uniformly distributed on the finite integer multisets characterizing them. Note that any uniform distribution on an integer multiset is equivalent to a rational distribution on an integer set. In the probabilistic sense, a collection $(X_1, X_2, \ldots, X_m)$ of independent discrete random variables, uniformly distributed on integer multisets, generates a probabilistic relation $Q = [q_{ij}]$, where

$$q_{ij} = \mathrm{Prob}\{X_i > X_j\} + \frac{1}{2}\,\mathrm{Prob}\{X_i = X_j\}\,. \tag{1}$$

The purpose of the present paper is to generalize the discrete dice model in such a way that other random variables than those that are uniformly distributed on finite integer multisets can be compared pairwisely in terms of a probabilistic relation. Moreover, we want to investigate the transitivity of that relation. Already in the case of the discrete dice model [3], the usual types of transitivity encountered in the context of probabilistic relations, such as, for instance, various types of stochastic transitivity and various types of T-transitivity (with T a t-norm), are not suited to describe in an accurate way the type of transitivity exhibited by the generated probabilistic relation and a framework is needed for harbouring a broader range of types of transitivity. The framework that proved to be the best suited is that of cycle-transitivity, recently established by the present authors [2]. In fact, the probabilistic relation of a discrete dice model exhibits a particular type of cycle-transitivity, called dice-transitivity [3]. Besides the study of the transitivity exhibited by the probabilistic relation of generalized dice models, we will also investigate the transitivity of probabilistic relations that are generated by random variables with parametric distribution functions. In particular, we will demonstrate that independent normal random variables generate moderately stochastic transitive probabilistic relations.

The outline of this paper is as follows. In Section 2 we introduce the concept of a generalized discrete or continuous dice model and show that its probabilistic relation can be interpreted as a graded alternative to the notion of stochastic dominance. In Section 3 we briefly review the framework of cycle-transitivity and the position held therein by dice-transitivity. Section 4 is concerned with the main theorem of the paper, which characterizes the type of transitivity exhibited by probabilistic relations of generalized dice models. All remaining sections are devoted to the study of the influence particular choices of random variables have on the transitivity of the generated probabilistic relation. In Section 5 we focus on random variables with shifted distributions, in Section 6 on random variables with distributions taken from certain parametric families of distributions, and in Section 7 special attention is paid to the case of normal random variables.

2 Generalized dice models

Clearly, definition (1) of the probabilistic relation Q of a discrete dice model can be immediately extended to compare arbitrary random variables. Indeed, any random vector $(X_1, X_2, \ldots, X_m)$ can, by means of the pairwise comparison of its components, serve as a source for generating a probabilistic relation.

Proposition 1 *For any random vector $(X_1, X_2, \ldots, X_m)$, the relation $Q = [q_{ij}]$ defined by*

$$q_{ij} = \mathrm{Prob}\{X_i > X_j\} + \frac{1}{2}\mathrm{Prob}\{X_i = X_j\} \tag{2}$$

is a probabilistic relation.

The definition of $Q = [q_{ij}]$ implies that the elements q_{ij} can be computed from the bivariate joint cumulative distribution functions (c.d.f.) F_{X_i,X_j} as follows

$$q_{ij} = \int\limits_{x>y} dF_{X_i,X_j}(x,y) + \frac{1}{2}\int\limits_{x=y} dF_{X_i,X_j}(x,y)\,. \tag{3}$$

Note that one should even not assume that the random variables are independent. In this paper, however, we will consider independent random variables only, and therefore bivariate distributions can always be factorized into univariate marginal distributions. If we want to further simplify (3), it is appropriate to distinguish between the following two cases.

Definition 2 *Let X_i, $i = 1, \ldots, m$, be independent discrete random variables, then the relation $Q = [q_{ij}]$ defined by*

$$q_{ij} = \sum_{k>l} p_{X_i}(k) p_{X_j}(l) + \frac{1}{2}\sum_{k} p_{X_i}(k) p_{X_j}(k)\,, \tag{4}$$

with p_{X_i} the marginal probability mass function of X_i, is a probabilistic relation. The discrete random variables together with the probabilistic relation they generate are called a generalized discrete dice model.

Definition 3 *Let X_i, $i = 1, \ldots, m$, be independent continuous random variables, then the relation $Q = [q_{ij}]$ defined by*

$$q_{ij} = \int_{-\infty}^{+\infty} f_{X_i}(x) \left(\int_{-\infty}^{x} f_{X_j}(y)\, dy \right) dx\,, \tag{5}$$

with f_{X_i} the marginal probability density function of X_i, is a probabilistic relation. The continuous random variables together with the probabilistic relation they generate are called a generalized continuous dice model.

Note that in the transition from the discrete to the continuous case, the second contribution to q_{ij} in (3) has disappeared in (5), since in the latter case $\mathrm{Prob}\{X_i = X_j\} = 0$. Of course, the information contained in the probabilistic relation is much richer than if for the pairwise comparison of X_i and X_j we would have used, for instance, only their expected values $\mathrm{E}[X_i]$ and $\mathrm{E}[X_j]$.

In the discussion of generalized dice models, we will maintain the terminology related to the original discrete dice model. A collection of dice will be kept as a metaphore for a collection of independent random variables. Two dice X_i and X_j, taken from a collection of dice, are compared in terms of the quantity q_{ij} for which it holds that $q_{ij} = 1 - q_{ji}$. If $q_{ij} > 1/2$, we still say that dice X_i wins from dice X_j, and if $q_{ij} = 1/2$, we say that both dice are statistically indifferent.

An alternative concept for comparing two random variables is that of stochastic dominance [8], which is particularly popular in financial mathematics.

Definition 4 *A random variable X with c.d.f. F_X stochastically dominates in first degree a random variable Y with c.d.f. F_Y, denoted as $X >_1 Y$, if for all real t it holds that $F_X(t) \leq F_Y(t)$, and the strict inequality holds for at least one t.*

The condition for first degree stochastic dominance is rather severe, as it requires that the graph of the function F_X lies beneath the graph of the function F_Y. The need to relax this condition has led to other types of stochastic dominance, such as second degree and third degree stochastic dominance. We will not go into more details here, since we just want to emphasize the following relationship between first degree stochastic dominance and the winning probabilities of a dice model.

Proposition 5 *For any two independent random variables X and Y it holds that $X >_1 Y$ implies $X >_s Y$.*

Proof: We give here the proof for continuous random variables, the proof for discrete ones being equally simple. Suppose that $X >_1 Y$, i.e. $F_X(z) \leq F_Y(z)$, for any $z \in \mathbb{R}$. Since F_X and F_Y are right-continuous functions, it holds that $F_X(z) < F_Y(z)$, for any $z \in I \subseteq \mathbb{R}$, for at least one non-degenerated interval I. Therefore, we obtain

$$\text{Prob}\{X > Y\} = \int_{-\infty}^{+\infty} f_X(x) F_Y(x) dx > \int_{-\infty}^{+\infty} f_X(x) F_X(x) dx = \frac{1}{2} .$$

□

The relation $>_s$ therefore generalizes first degree stochastic dominance $>_1$. As the probabilistic relation of a dice model is a graded version of the crisp relation $>_s$, we can therefore interpret this relation as a graded alternative to first degree stochastic dominance.

3 Cycle-transitivity

3.1 Definition of cycle-transitivity

In the framework of cycle-transitivity [2], for a probabilistic relation $Q = [q_{ij}]$, the quantities

$$\alpha_{ijk} = \min(q_{ij}, q_{jk}, q_{ki}) \, , \;\; \beta_{ijk} = \text{med}(q_{ij}, q_{jk}, q_{ki}) \, , \;\; \gamma_{ijk} = \max(q_{ij}, q_{jk}, q_{ki}) \, ,$$

are defined for all (i, j, k). Obviously, $\alpha_{ijk} \leq \beta_{ijk} \leq \gamma_{ijk}$. Also, the notation $\Delta = \{(x, y, z) \in [0, 1]^3 \,|\, x \leq y \leq z\}$ will be used.

Definition 6 *A function $U : \Delta \to \mathbb{R}$ is called an upper bound function if it satisfies:*

(i) $U(0, 0, 1) \geq 0$ and $U(0, 1, 1) \geq 1$;
(ii) for any $(\alpha, \beta, \gamma) \in \Delta$:

$$U(\alpha, \beta, \gamma) + U(1 - \gamma, 1 - \beta, 1 - \alpha) \geq 1 \, .$$

The function $L : \Delta \to \mathbb{R}$ defined by

$$L(\alpha, \beta, \gamma) = 1 - U(1 - \gamma, 1 - \beta, 1 - \alpha)$$

is called the *dual lower bound function* of a given upper bound function U.

Definition 7 *A probabilistic relation $Q = [q_{ij}]$ is called cycle-transitive w.r.t. an upper bound function U, if for all (i, j, k) it holds that*

$$L(\alpha_{ijk}, \beta_{ijk}, \gamma_{ijk}) \leq \alpha_{ijk} + \beta_{ijk} + \gamma_{ijk} - 1 \leq U(\alpha_{ijk}, \beta_{ijk}, \gamma_{ijk}), \tag{6}$$

where L is the dual lower bound function of U.

If (6) holds for some (i, j, k), then due to the built-in duality, it also holds for all permutations of (i, j, k). On the other hand, this duality implies that it is sufficient to verify only the right-hand inequality (or equivalently, only the left-hand inequality) for two permutations of (i, j, k) that are not cyclic permutations of one another, e.g. (i, j, k) and (k, j, i). When the lower bound function equals the upper bound function, i.e. $L(a, b, c) = U(a, b, c)$ for all $(a, b, c) \in \Delta$ (in which case the inequalities in (6) become equalities), we say that the function U is *self-dual.*

Note that a value of $U(\alpha, \beta, \gamma)$ equal to 2 will often be used to express that for the given values there is no restriction at all (indeed, $\alpha + \beta + \gamma - 1$ is always bounded by 2). The above definition implies that if a probabilistic relation Q is cycle-transitive w.r.t. U_1 and $U_1(a, b, c) \leq U_2(a, b, c)$ for all $(a, b, c) \in \Delta$, then Q is cycle-transitive w.r.t. U_2. It is clear that $U_1 \leq U_2$ is not a necessary condition for the latter implication to hold. Two upper bound functions U_1 and U_2 will be called *equivalent* if for any $(\alpha, \beta, \gamma) \in \Delta$ it holds that

$$\alpha + \beta + \gamma - 1 \leq U_1(\alpha, \beta, \gamma)$$

is equivalent to

$$\alpha + \beta + \gamma - 1 \leq U_2(\alpha, \beta, \gamma).$$

For instance, suppose that the inequality $\alpha + \beta + \gamma - 1 \leq U_1(\alpha, \beta, \gamma)$ can be rewritten as

$$\alpha \leq h(\beta, \gamma),$$

then an equivalent upper bound function U_2 is given by

$$U_2(\alpha, \beta, \gamma) = \beta + \gamma - 1 + h(\beta, \gamma).$$

In this way, it is often possible to find an equivalent upper bound function in only two of the variables α, β and γ.

Cycle-transitivity includes as special cases T-transitivity and all known types of g-stochastic transitivity.

Let us recall that a binary operation $T : [0,1]^2 \to [0,1]$ is called a *t-norm* if it is increasing, associative, commutative and possesses 1 as neutral element [9]. A $[0,1]$-valued relation R on a set of alternatives A is called T-transitive [6] if for any $(a,b,c) \in A^3$ it holds that $T(R(a,b), R(b,c)) \leq R(a,c)$. The following proposition [2] shows how T-transitivity fits into the framework of cycle-transitivity in case the t-norm T is 1-Lipschitz continuous (for short, 1-Lipschitz), which means that for all $(x,y,z) \in [0,1]^3$ it holds that $|T(x,y) - T(x,z)| \leq |y - z|$.

Proposition 8 *Let T be a 1-Lipschitz t-norm. A probabilistic relation is T-transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_T defined by*

$$U_T(\alpha, \beta, \gamma) = \alpha + \beta - T(\alpha, \beta)\,. \tag{7}$$

Note that 1-Lipschitz t-norms can also be regarded as associative and commutative copulas. Copulas play a predominant role in expressing bivariate cumulative distribution functions in terms of univariate marginal distribution functions [11]. The following special cases of 1-Lipschitz t-norms are of particular interest:

(i) $T_{\mathbf{M}}(x,y) = \min(x,y)$ with $U_{\mathbf{M}}(\alpha,\beta,\gamma) = \beta$;
(ii) $T_{\mathbf{P}}(x,y) = xy$ with $U_{\mathbf{P}}(\alpha,\beta,\gamma) = \alpha + \beta - \alpha\beta$;
(iii) $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$ with $U_{\mathbf{L}}(\alpha,\beta,\gamma) = \min(\alpha+\beta,1)$.
An equivalent upper bound function is given by $U'_{\mathbf{L}}(\alpha,\beta,\gamma) = 1$.

In the literature one finds various types of stochastic transitivity [1,10]. They can, however, be regarded as special cases of a generic type of stochastic transitivity, which we have called g-stochastic transitivity. Let g be a commutative, increasing $[1/2,1]^2 \to [1/2,1]$ mapping. A probabilistic relation Q on A is called g-stochastic transitive if for any $(a,b,c) \in A^3$ it holds that

$$(Q(a,b) \geq 1/2 \ \wedge \ Q(b,c) \geq 1/2) \ \Rightarrow \ Q(a,c) \geq g(Q(a,b), Q(b,c))\,.$$

In [2], we have proven the following proposition.

Proposition 9 *Let g be a commutative, increasing $[1/2,1]^2 \to [1/2,1]$ mapping such that $g(1/2,x) \leq x$ for any $x \in [1/2,1]$. A probabilistic relation Q on A is g-stochastic transitive if and only if it is cycle-transitive w.r.t. the upper* *bound function U_g defined by*

$$U_g(\alpha,\beta,\gamma) = \begin{cases} \beta+\gamma-g(\beta,\gamma) & \text{, if } \beta \geq 1/2 \wedge \alpha < 1/2\,, \\ 1/2 & \text{, if } \alpha \geq 1/2\,, \\ 2 & \text{, if } \beta < 1/2\,. \end{cases} \tag{8}$$

We obtain as special cases (only mentioning the function g):

(i) strong stochastic transitivity: $g_{ss}(\beta,\gamma) = \max(\beta,\gamma) = \gamma$;
(ii) moderate stochastic transitivity: $g_{ms}(\beta,\gamma) = \min(\beta,\gamma) = \beta$;
(iii) weak stochastic transitivity: $g_{ws}(\beta,\gamma) = 1/2$.

In our study of the probabilistic relations of dice models, a type of transitivity which can neither be classified as a type of T-transitivity, nor as a type of g-stochastic transitivity, has proven to play a predominant role and this new type of transitivity has been called dice-transitivity.

Definition 10 *Cycle-transitivity w.r.t. the upper bound function U_D defined by*

$$U_D(\alpha,\beta,\gamma) = \beta+\gamma-\beta\gamma\,, \tag{9}$$

is called dice-transitivity.

One easily verifies that an equivalent upper bound function for U_D is given by

$$U_D(\alpha,\beta,\gamma) = \begin{cases} \beta+\gamma-\beta\gamma & \text{, if } \beta \geq 1/2\,, \\ 2 & \text{, if } \beta < 1/2\,, \end{cases}$$

since for $\beta < 1/2$ the inequality $\alpha - 1 \leq -\beta\gamma$ is trivially fulfilled. As it holds that $U_{\mathbf{P}} \leq U_D \leq U_{\mathbf{L}}$ and also $U_{ms} \leq U_D$ (with $U_{ms}(\alpha,\beta,\gamma) = \beta + \gamma - g_{ms}(\beta,\gamma) = \gamma$ for $\beta \geq 1/2$ and $\alpha < 1/2$), dice-transitivity can be situated between $T_{\mathbf{P}}$-transitivity and $T_{\mathbf{L}}$-transitivity, and also between moderate stochastic transitivity and $T_{\mathbf{L}}$-transitivity.

3.2 Self-dual upper bound functions

As stated above, any upper bound function U that coincides with its corresponding lower bound function L is called a self-dual upper bound function.

The following proposition yields a way to construct self-dual upper bound functions [2].

Proposition 11 *Let h be a commutative, increasing $[1/2,1]^2 \to [1/2,1]$ mapping with neutral element $1/2$. It then holds that any $\Delta \to \mathbb{R}$ function U of the form*

$$U_h^s(\alpha,\beta,\gamma) = \begin{cases} \beta+\gamma-h(\beta,\gamma) & \text{, if } \beta \geq 1/2\,, \\ \alpha+\beta-1+h(1-\beta,1-\alpha) & \text{, if } \beta < 1/2\,, \end{cases} \tag{10}$$

is a self-dual upper bound function.

One easily verifies that $U_h^s \leq U_{ss}$ (with $U_{ss}(\alpha,\beta,\gamma)=\gamma$ for $\beta \geq 1/2$), and hence cycle-transitivity w.r.t. U_h^s implies strong stochastic transitivity.

Example 12 The upper bound function $U_{\mathbf{M}}(\alpha,\beta,\gamma)=\beta$, which characterizes $T_{\mathbf{M}}$-transitivity, is a self-dual upper bound function of the form (10) with $h=\max$. ▷

Example 13 Another example of a self-dual upper bound function is the function U_E defined by

$$U_E(\alpha,\beta,\gamma) = \alpha\beta+\alpha\gamma+\beta\gamma-2\alpha\beta\gamma\,. \tag{11}$$

Solving α (resp. γ) from the equation $\alpha+\beta+\gamma-1=\alpha\beta+\alpha\gamma+\beta\gamma-2\alpha\beta\gamma$ and substituting the solution in the expression for $U_E(\alpha,\beta,\gamma)$ in case $\beta \geq 1/2$ (resp. $\beta < 1/2$), we obtain the equivalent self-dual upper bound function

$$U_E'(\alpha,\beta,\gamma) = \begin{cases} \beta+\gamma-\dfrac{\beta\gamma}{\beta\gamma+(1-\beta)(1-\gamma)} & \text{, if } \beta \geq 1/2\,, \\ \alpha+\beta-1+\dfrac{(1-\alpha)(1-\beta)}{\alpha\beta+(1-\alpha)(1-\beta)} & \text{, if } \beta < 1/2\,, \end{cases} \tag{12}$$

which is of the form (10) with h defined by

$$h(x,y) = \frac{x\,y}{x\,y+(1-x)(1-y)}\,. \tag{13}$$

▷

Note that cycle-transitivity w.r.t. U_E of a probabilistic relation $Q=[q_{ij}]$ can also be expressed as

$$\alpha_{ijk}+\beta_{ijk}+\gamma_{ijk}-1 = \alpha_{ijk}\beta_{ijk}+\alpha_{ijk}\gamma_{ijk}+\beta_{ijk}\gamma_{ijk}-2\alpha_{ijk}\beta_{ijk}\gamma_{ijk}\,,$$

or, equivalently, as

$$\alpha_{ijk}\beta_{ijk}\gamma_{ijk} = (1-\alpha_{ijk})(1-\beta_{ijk})(1-\gamma_{ijk})\,.$$

Cycle-transitivity w.r.t. the upper bound function U_E is therefore equivalent to the concept of multiplicative transitivity recalled below [15]. Note that the cycle-transitive version is more appropriate as it avoids division by zero.

Definition 14 *A probabilistic relation $Q = [q_{ij}]$ is called multiplicatively transitive if for all (i, j, k) it holds that*

$$\frac{q_{ik}}{q_{ki}} = \frac{q_{ij}}{q_{ji}}\frac{q_{jk}}{q_{kj}}\,. \tag{14}$$

As self-dual upper bound functions typically turn inequalities into equalities, the following proposition does not come as a surprise. It shows that cycle-transitivity w.r.t. an upper bound function of type (10) can be seen as a variant of g-stochastic transitivity.

Proposition 15 *A probabilistic relation Q on A is cycle-transitive w.r.t. a self-dual upper bound function of type U_h^s if and only if for any $(a, b, c) \in A^3$ it holds that*

$$(Q(a,b) \geq 1/2 \ \wedge\ Q(b,c) \geq 1/2) \ \Rightarrow\ Q(a,c) = h(Q(a,b), Q(b,c))\,. \tag{15}$$

The probabilistic relation Q will also be called isostochastic transitive w.r.t. h, or shortly, h-isostochastic transitive.

In particular, a reciprocal relation Q is $T_{\mathbf{M}}$-transitive if and only if

$$(Q(a,b) \geq 1/2 \ \wedge\ Q(b,c) \geq 1/2) \ \Rightarrow\ Q(a,c) = \max(Q(a,b), Q(b,c))\,,$$

for any $(a, b, c) \in A^3$.

Note that the properties imposed on h in Proposition 11 are very close to the defining properties of t-conorms. Let us recall that a binary operation $S : [0,1]^2 \to [0,1]$ is a t-conorm if it is increasing, associative, commutative and possesses 0 as neutral element [9]. Indeed, although associativity is not explicitly required for h, it follows quite naturally. Consider for instance an h-isostochastic transitive probabilistic relation Q such that $Q(a,b) \geq 1/2$, $Q(b,c) \geq 1/2$ and $Q(c,d) \geq 1/2$. It then holds that

$$Q(a,d) = h(Q(a,b), Q(b,d)) = h(Q(a,b), h(Q(b,c), Q(c,d)))$$

and

$$Q(a,d) = h(Q(a,c), Q(c,d)) = h(h(Q(a,b), Q(b,c)), Q(c,d))\,,$$

whence at least for the triplet $(Q(a,b), Q(b,c), Q(c,d))$ the function h is associative.

Adding (full) associativity makes h into a t-conorm on $[1/2, 1]$, or after appropriate rescaling, into a usual t-conorm on $[0, 1]$.

Proposition 16 *If h is an associative, commutative, increasing $[1/2, 1]^2 \to [1/2, 1]$ mapping with neutral element $1/2$, then the $[0, 1]^2 \to [0, 1]$ mapping S_h defined by*

$$S_h(x, y) = 2h\left(\frac{1+x}{2}, \frac{1+y}{2}\right) - 1$$

is a t-conorm.

The two examples of self-dual upper bound functions given above fall into the latter category. For the self-dual upper bound function U'_E in (12), the associated t-conorm S_E is given by

$$S_E(x, y) = \frac{x+y}{1+xy}\,, \tag{16}$$

which is the Hamacher t-conorm $S_2^{\mathbf{H}}$ with parameter value 2 [9].

This t-conorm is a member of the well-known class of strict t-conorms which are of the form

$$S(x, y) = g^{-1}(g(x) + g(y))\,,$$

with g an additive generator, i.e. a strictly increasing and continuous $[0, 1] \to [0, +\infty]$ function that satisfies $g(0) = 0$ (see e.g. [9]).

4 Transitivity of generalized dice models

In previous work [3], we have proven the following remarkable theorem.

Theorem 17 *The probabilistic relation of a discrete dice model is dice-transitive.*

In this section, we prove one of the main results of this paper, namely that the probabilistic relation of any generalized dice model, whether discrete or continuous, is dice-transitive. More precisely, we proceed in two distinct steps: firstly, the discrete dice model is generalized to cover the case of arbitrary discrete random variables, and, secondly, the generalization to arbitrary continuous random variables is considered.

Theorem 18 *The probabilistic relation of a generalized discrete dice model is dice-transitive.*

Proof: First, we want to emphasize that the introduction of negative integers in the multisets of a discrete dice model does not alter the transitivity. Let X_k be a random variable of a generalized discrete dice model, and let I_n with $n > 0$ denote the following set of $2n+1$ integers: $I_n = \{i \in \mathbb{Z} \mid -n \leq i \leq n\}$. We now approximate the random variable X_k by a random variable $X_k^{(n)}$ which takes values in I_n with rational probabilities only, in such a way that:

$$
\begin{aligned}
&p_{X_k^{(n)}}(-n) \in \mathbb{Q} \ \wedge \ 0 \leq \mathrm{Prob}\{X_k \leq -n\} - p_{X_k^{(n)}}(-n) < \frac{1}{n^2}\,,\\
&p_{X_k^{(n)}}(j) \in \mathbb{Q} \ \wedge \ 0 \leq \mathrm{Prob}\{X_k = j\} - p_{X_k^{(n)}}(j) < \frac{1}{n^2}\,,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall j \in I_m \setminus \{-n, n\}\,,\\
&p_{X_k^{(n)}}(n) = 1 - \sum_{i=-n}^{n-1} p_{X_k^{(n)}}(i)\,.
\end{aligned}
$$

It is clear that such an approximation always exists, since the set of rationals $\mathbb{Q}$ is dense in the set of reals $\mathbb{R}$. From the above inequalities, it also follows that

$$
p_{X_k^{(n)}}(n) - \mathrm{Prob}\{X_k \geq n\} < \frac{2}{n}\,.
$$

Since we can take n as large as we like, the generalized discrete dice model can be approximated with arbitrary precision by a discrete dice model in which the dice have a finite number of faces, each face containing one integer, and the probability of a particular face showing up in a random roll of the dice being for each face a rational number. Bringing all rational probabilities to a (least) common denominator, it suffices to duplicate, depending on the numerator values, each face a number of times in order to obtain an equivalent discrete dice model. The result of all these operations is that any generalized discrete dice model can be approximated with arbitrary precision by a discrete dice model. In particular, the probabilistic relation $Q = [q_{ij}]$ of a generalized discrete dice model can be approximated with arbitrary precision by the probabilistic relation $Q^{(\epsilon)} = [q_{ij}^{(\epsilon)}]$ of a discrete dice model, where for all

$\epsilon > 0$ and all (i, j) it holds that $|q_{ij} - q_{ij}^{(\epsilon)}| < \epsilon$. Since all Q^ϵ are dice-transitive and $Q = \lim_{\epsilon \to 0} Q^{(\epsilon)}$, also Q is dice-transitive. $\square$

We now execute the second step mentioned before, by considering continuous dice models.

Theorem 19 *The probabilistic relation of a generalized continuous dice model is dice-transitive.*

Proof: Let X_k be a random variable of a generalized continuous dice model with probability density function f_{X_k}. We partition $\mathbb{R}$ into an infinite number of segments: $\mathbb{R} = \cup_{n=-\infty}^{+\infty} \delta_n$, with $\delta_n = [n\delta, (n+1)\delta[$ and arbitrary $\delta > 0$. We approximate the continuous random variable X_k by a discrete random variable $X_k^{(\delta)}$ with probability mass function $p_{X_k}^{(\delta)}$:

$$p_{X_k}^{(\delta)}(i) = \int_{i\delta}^{(i+1)\delta} f_{X_k}(x)\, dx\,, \quad i \in \mathbb{Z}\,.$$

Since δ can be chosen as small as one likes, the generalized continuous dice model can be approximated with arbitrary precision by a generalized discrete dice model, and, in particular, its probabilistic relation Q can be (elementwise) approximated by the dice-transitive probabilistic relation $Q^{(\delta)}$ of a generalized discrete dice model. Since $\lim_{\delta \to 0} Q^{(\delta)} = Q$, and since dice-transitivity is expressed through inequalities, Q inherits the transitivity of the approximating relations $Q^{(\delta)}$, whence Q is dice-transitive. $\square$

To conclude this section, let us reformulate the main result as follows. The discrete dice model with random variables that are uniformly distributed on integer multisets, is as far as the transitivity of the generated probabilistic relation is concerned, a generic model, in the sense that all generalized dice models generate dice-transitive probabilistic relations.

Of course, if the random variables of a generalized dice model possess distribution functions that obey certain constraints, then it can happen that the transitivity of the generated probabilistic relation is of a stronger type than dice-transitivity. In the remaining sections, we will discuss certain of these constraints and their influence on the type of transitivity.

5 Dice with shifted distributions

As a first example of generalized dice models in which certain constraints are imposed on the distribution functions of the random variables, we consider the case where these random variables possess cumulative distribution functions that are translated copies of a generic cumulative distribution function F_X. We will investigate the transitivity of the probabilistic relations generated by such restricted dice models and the notion of isostochastic transitivity will naturally come to the foreground.

Proposition 20 *Let the c.d.f. F_{X_i} of the independent random variables X_i, $i = 1, \ldots, m$, of a generalized dice model be arbitrary translations of the same c.d.f. F_X, i.e. $F_{X_i}(x) = F_X(x - t_i)$ for all i with arbitrary real t_i. If for all $u \neq v$ for which the equality*

$$\int_{-\infty}^{+\infty} F_X(x-u)\, dF_X(x) = \int_{-\infty}^{+\infty} F_X(x-v)\, dF_X(x) \tag{17}$$

holds, the integrals are either both 0 or both 1, then the probabilistic relation generated by the random variables is isostochastic transitive w.r.t. a function h that solely depends upon the generic c.d.f. F_X.

Proof: We can assume without loss of generality that the indices of three random variables X_i, X_j, X_k are such that $q_{ij} \geq 1/2$ and $q_{jk} \geq 1/2$. The value of q_{ij} is computed as follows

$$q_{ij} = \int_{-\infty}^{+\infty} F_X(x-t_j)\, dF_X(x-t_i) = \int_{-\infty}^{+\infty} F_X(x+t_i-t_j)\, dF_X(x)\,.$$

Since F_X is non-decreasing and the last integral is equal to $1/2$ when $t_i = t_j$, it is clear that $q_{ij} \geq 1/2$ implies $t_i \geq t_j$. Similarly, it holds that

$$q_{jk} = \int_{-\infty}^{+\infty} F_X(x+t_j-t_k)\, dF_X(x)\,,$$

with $t_j \geq t_k$. Finally,

$$q_{ik} = \int_{-\infty}^{+\infty} F_X(x+t_i-t_k)\, dF_X(x)\,,$$

and since $t_i - t_k = (t_i - t_j) + (t_j - t_k)$, we immediately obtain that

$$q_{ik} \geq \max(q_{ij}, q_{jk}) \geq \frac{1}{2}\,. \tag{18}$$

Let us first assume that $q_{ij} \neq 1$ and $q_{jk} \neq 1$. Then, due to (17), the differences $t_i - t_j$ and $t_j - t_k$ are unique, and so is their sum $t_i - t_k$. If $q_{ij} = 1$ or $q_{jk} = 1$, then, according to (18), also $q_{ik} = 1$. This proves that q_{ik} is a function of q_{ij} and q_{jk} on $[1/2, 1]^2$, which we denote as $q_{ik} = h(q_{ij}, q_{jk})$ with h a $[1/2, 1]^2 \to [1/2, 1]$ function solely depending upon F_X. It is easy to verify that h is increasing and has $1/2$ as neutral element. For instance, if $q_{ij} = 1/2$ then condition (17) implies that $t_i = t_j$, whence $q_{ik} = h(1/2, q_{jk}) = q_{jk}$. Furthermore, h is symmetric and since $q_{ki} = 1 - q_{ik} \leq 1/2$, we can rewrite, using previously introduced notations, the functional relationship as $1 - \alpha_{ijk} = h(\beta_{ijk}, \gamma_{ijk})$ if $\beta_{ijk} \geq 1/2$, or equivalently

$$\alpha_{ijk} + \beta_{ijk} + \gamma_{ijk} - 1 = \beta_{ijk} + \gamma_{ijk} - h(\beta_{ijk}, \gamma_{ijk}) \quad \text{if } \beta_{ijk} \geq 1/2\,.$$

Since the above equality holds for all (i, j, k) for which $\beta_{ijk} \geq 1/2$, it follows that the probabilistic relation Q is cycle-transitive w.r.t. the self-dual function U_h^s defined in (10). Hence, according to the terminology introduced in Proposition 15, the probabilistic relation Q is h-isostochastic transitive. □

Note that $q_{ij} = 1$ implies that $F_X(x+t_i-t_j) = 1$ for all x for which $dF_X(x) \neq 0$. Hence, $q_{ij} = 1$ implies that $x+t_i-t_j \geq \tau_u$ should be satisfied for all $x \in [\tau_l, \tau_u]$, where τ_l and τ_u are the lower and upper bounds of the support of dF_X, or equivalently, $t_i - t_j \geq \tau_u - \tau_l = \tau$, where τ is the range of this support. This can therefore only occur if the distribution of X has finite support.

Finally, it must be emphasized that condition (17) is not only a sufficient but also a necessary condition for the h-isostochastic transitivity. However, in a continuous dice model, it is sufficient that the distribution of X has either infinite support or has as finite support a single interval. In a discrete dice model, it is sufficient that the probability mass function is strictly positive on a single interval of integers and zero elsewhere.

Example 21 As a first example of a dice model with shifted distributions, let us consider the case of the exponential distribution with parameter λ, i.e. $F_X(x) = 1 - \exp(-\lambda x)$. Let us assume that the translational parameters for the three random variables X_i, X_j, X_k are such that $t_i \geq t_j \geq t_k$. We compute:

$$q_{ij} = \text{Prob}\{X_i > X_j\} = \int_{t_i}^{+\infty} \lambda e^{-\lambda(x-t_i)}[1 - e^{-\lambda(x-t_j)}]\, dx = 1 - \frac{1}{2}e^{-\lambda(t_i-t_j)}\,,$$

from which it follows that $\exp(-\lambda(t_i - t_j)) = 2(1 - q_{ij})$. Similarly, it holds that $\exp(-\lambda(t_j - t_k)) = 2(1 - q_{jk})$. This leads to

$$q_{ik} = 1 - \frac{1}{2}e^{-\lambda(t_i-t_k)} = 1 - \frac{1}{2}e^{-\lambda(t_i-t_j)}e^{-\lambda(t_j-t_k)} = 1 - 2(1 - q_{ij})(1 - q_{jk})\,.$$

Since $t_i \geq t_j \geq t_k$, it holds that $q_{ij} \geq 1/2$, $q_{jk} \geq 1/2$ and $q_{ki} \leq 1/2$, and the foregoing expression can be rewritten as

$$1 - \alpha_{ijk} = 1 - 2(1 - \beta_{ijk})(1 - \gamma_{ijk})\,.$$

It then follows that Q is isostochastic transitive w.r.t. the function h defined by

$$h(x, y) = 1 - 2(1 - x)(1 - y)\,. \tag{19}$$

Using Proposition 16, we obtain the associated t-conorm S_h as

$$S_h(x, y) = x + y - xy\,,$$

which is the well-known probabilistic sum. ▷

Example 22 As a second example, we consider the Gumbel distribution $G(\mu, \eta)$ as the generic distribution for a collection of shifted random variables. Let us recall that a continuous random variable X on $\mathbb{R}$ is said to be Gumbel-distributed with parameters μ and η, if it holds that:

$$f_X(x) = \mu e^{-\mu(x-\eta)} e^{-e^{-\mu(x-\eta)}}\,, \tag{20}$$

for any $x \in \mathbb{R}$. The corresponding c.d.f. is then given by

$$F_X(x) = e^{-e^{-\mu(x-\eta)}}\,.$$

The random variable X has expected value $\eta + C/\mu$ and variance $\pi^2/(6\mu^2)$, with C the Euler-Masceroni constant. It is known that if $X_1 \stackrel{d}{=} G(\mu, \eta_1)$ and $X_2 \stackrel{d}{=} G(\mu, \eta_2)$ are two independent Gumbel-distributed random variables with same variance (same μ), then $\max(X_1, X_2)$ is Gumbel-distributed with the same μ and with parameter $\eta = \ln(e^{\mu\eta_1} + e^{\mu\eta_2})/\mu$, whereas $X_1 - X_2$ is a random variable that has the logistic distribution, i.e.:

$$F_{X_1 - X_2}(x) = \frac{1}{1 + e^{\mu(\eta_2 - \eta_1 - x)}}\,. \tag{21}$$

Let us assume that X_i, X_j, X_k are three random variables with distributions shifted by t_i, t_j, t_k from the generic Gumbel distribution $G(\mu, \eta)$. Then

$$q_{ij} = 1 - F_{X_i - X_j}(0) = \frac{e^{\mu(\eta_j - \eta_i)}}{1 + e^{\mu(\eta_j - \eta_i)}} = \frac{e^{\mu\eta_j}}{e^{\mu\eta_i} + e^{\mu\eta_j}}\,.$$

Using the short notation $\lambda_i = \exp(\mu\,\eta_i)$, we obtain

$$q_{ij} = \lambda_i \int_0^{+\infty} e^{-(\lambda_i+\lambda_j)z}\,dz = \frac{\lambda_i}{\lambda_i+\lambda_j}\,,$$

from which we immediately obtain that $q_{ij}/q_{ji} = \lambda_i/\lambda_j$. Since obviously equality (14) is satisfied for all (i, j, k), the probabilistic relation Q is multiplicatively transitive, or, equivalently, isostochastic transitive w.r.t. the function h defined in (13). Let us recall that the t-conorm S_h associated to it, is the Hamacher t-conorm $S_2^{\mathbf{H}}$ with parameter value 2 defined in (16). ▷

6 Dice models with parametric random variables

6.1 Families considered

Dice-transitivity is the generic type of transitivity shared by the probabilistic relation generated by a collection of arbitrary discrete or continuous independent random variables. Clearly, stronger types of transitivity might be obtained when one restricts the distributions of the random variables to particular families of distributions, such as certain standard parametric families.

In particular, we will investigate continuous random variables with probability density functions taken from a one-parameter family of density functions. These families and density functions are listed in Table 1 (the variable parameter in all cases being λ, while the other parameters are treated as constants). In the case of normal distributions, for example, we only consider the one-parameter subfamily of normal distributions with varying expected value and constant variance.

6.2 Examples of multiplicative transitivity

6.2.1 Exponentially distributed dice

Let us consider the case of exponentially distributed dice, i.e. $X_i \stackrel{d}{=} E(\lambda_i)$. It then holds that

$$q_{ij} = \int_0^{+\infty} \lambda_i e^{-\lambda_i x} dx \int_0^{x} \lambda_j e^{-\lambda_j y} dy = \frac{\lambda_i}{\lambda_i+\lambda_j}\,,$$

Table 1
Parametric families of continuous distributions.

Name	Density function $f(x)$		
Exponential	$\lambda e^{-\lambda x}$	$\lambda > 0$	$x \in [0, +\infty[$
Beta	$\lambda x^{(\lambda-1)}$	$\lambda > 0$	$x \in [0, 1]$
Pareto	$\lambda x^{-(\lambda+1)}$	$\lambda > 0$	$x \in [1, +\infty[$
Gumbel	$\mu e^{-\mu(x-\lambda)} e^{-e^{-\mu(x-\lambda)}}$	$\lambda \in \mathbb{R},\ \mu > 0$	$x \in]-\infty, +\infty[$
Uniform	$1/a$	$\lambda \in \mathbb{R},\ a > 0$	$x \in [\lambda, \lambda + a]$
Laplace	$e^{-\lvert x-\lambda \rvert/\mu)}/(2\mu)$	$\lambda \in \mathbb{R},\ \mu > 0$	$x \in]-\infty, +\infty[$
Normal	$e^{-(x-\lambda)^2/(2\sigma^2)}/\sqrt{2\pi\sigma^2}$	$\lambda \in \mathbb{R}, \sigma > 0$	$x \in]-\infty, +\infty[$

and it follows that $q_{ij}/q_{ji} = \lambda_i/\lambda_j$, which shows that Q is again multiplicatively transitive.

It is worthwhile to remark that the same transitivity property holds for the probabilistic relation Q generated by independent discrete random variables $X_i \stackrel{d}{=} G(p_i)$ that are geometrically distributed (i.e. $p_{X_i}(k) = p_i(1-p_i)^{k-1}\,, 0 < p_i < 1\,, k \geq 1$). Indeed, taking into consideration (4), we compute:

$$\begin{aligned} q_{ij} &= \sum_{k=1}^{+\infty}(1-p_j)^{k-1}p_j \sum_{l=k+1}^{+\infty}(1-p_i)^{l-1}p_i + \frac{1}{2}\sum_{k=1}^{+\infty}(1-p_i)^{k-1}(1-p_j)^{k-1}p_ip_j \\ &= \frac{p_j(1-p_i/2)}{p_i+p_j-p_ip_j}\,, \end{aligned}$$

and one can easily verify that the equality $q_{ij}q_{jk}q_{ki} = (1-q_{ij})(1-q_{jk})(1-q_{ki})$ again holds. It is, after all, not so surprising that geometric distributions yield the same type of transitivity as exponential distributions, since the former can be regarded as a discretization of the latter.

6.2.2 Dice with a power-law distribution

The one-parameter power-law distributions mentioned in Table 1 form a subfamily of the family of Beta-distributions as well as of the family of Pareto-distributions, the former ones having finite support, the latter ones having infinite support. We leave it to the reader to verify that in both cases

$$q_{ij} = \frac{\lambda_i}{\lambda_i + \lambda_j}\,,$$

which allows us to conclude that the generated probabilistic relation Q is again multiplicatively transitive.

6.2.3 Gumbel-distributed dice

In Example 22, we have already introduced the two-parameter family of Gumbel distributions. By choosing $X_i \stackrel{d}{=} G(\mu, \lambda_i)$, the distribution of X_i can be regarded as the generic distribution $G(\mu, 0)$ shifted by λ_i. Hence, the result of Example 22 immediately applies, namely, the generated probabilistic relation Q is again multiplicatively transitive.

6.3 Other examples of isostochastic transitivity

Note that the remaining one-parameter families of distributions from Table 1 all concern distributions that for varying λ can be regarded as shifted versions of a single generic distribution. All these cases could therefore equally well have been treated before as examples of dice with shifted distributions, and moreover, we can already state, since the conditions of Proposition 20 are always fulfilled, that these families of distributions all generate a probabilistic relation that is h-isostochastic transitive, and hence also strongly stochastic transitive. It remains to characterize that function h for each of these families.

6.3.1 Dice with a unimodal uniform distribution

Let us consider independent random variables $X_i \stackrel{d}{=} U[\lambda_i, \lambda_i + a]$ and let us further assume without loss of generality that X_i, X_j, X_k are three such random variables for which it holds that $\lambda_i \geq \lambda_j \geq \lambda_k$. If $\lambda_i \geq \lambda_j + a$ then $q_{ij} = 1$ and if $\lambda_j \leq \lambda_i < \lambda_j + a$, then by straightforward computation we obtain

$$q_{ij} = 1 - \frac{(a + \lambda_j - \lambda_i)^2}{2a^2} \,.$$

Note that $\lambda_i \geq \lambda_j$ implies that $q_{ij} \geq 1/2$. Introducing the short notation $s_{ij} = \max(a+\lambda_j-\lambda_i, 0)$, it follows that if $\lambda_i \geq \lambda_j$ then $q_{ij} = 1-s_{ij}^2/(2a^2)$. Similarly, since $\lambda_j \geq \lambda_k$, it holds that $q_{jk} = 1-s_{jk}^2/(2a^2)$ and $q_{ik} = 1-s_{ik}^2/(2a^2)$. Solving s_{ij} (resp. s_{jk}) in terms of q_{ij} (resp. q_{jk}), we find $s_{ij} = a\sqrt{2(1-q_{ij})}$ (resp. $s_{jk} = a\sqrt{2(1-q_{jk})}$). Since furthermore

$$s_{ik} = \max((a + \lambda_k - \lambda_j) + (a + \lambda_j - \lambda_i) - a, 0) = \max(s_{ij} + s_{jk} - a, 0) \,,$$

we obtain

$$q_{ik} = 1 - \frac{\left(\max(a\sqrt{2(1-q_{ij})} + a\sqrt{2(1-q_{jk})} - a, 0)\right)^2}{2a^2} \,,$$

which proves that the generated probabilistic relation Q is isostochastic transitive w.r.t. the function h defined by

$$h(x,y) = 1 - \frac{1}{2}\left(\max(\sqrt{2(1-x)} + \sqrt{2(1-y)} - 1, 0)\right)^2 \,.$$

The associated t-conorm S_h, given by

$$S_h(x,y) = 1 - \left(\max(\sqrt{1-x} + \sqrt{1-y} - 1, 0)\right)^2 \,,$$

is nothing else but the Schweizer-Sklar t-conorm $S^{\mathbf{SS}}_{1/2}$ with parameter value 1/2 [9].

6.3.2 Laplace-distributed dice (with constant variance)

Let $X_i \stackrel{d}{=} \mathrm{Lap}(\lambda_i, \mu_i)$ be Laplace-distributed random variables with parameters $\lambda_i, \mu_i > 0$, namely $f_{X_i}(x) = \exp(-|x-\lambda_i|/\mu_i)/(2\mu_i)$, then a straightforward computation leads to

$$q_{ij} = \begin{cases} 1 - \dfrac{1}{2(\mu_i^2 - \mu_j^2)}[\mu_i^2 e^{-(\lambda_i-\lambda_j)/\mu_i} - \mu_j^2 e^{-(\lambda_i-\lambda_j)/\mu_j}] & , \text{ if } \lambda_i \geq \lambda_j \,, \\ \dfrac{1}{2(\mu_i^2 - \mu_j^2)}[\mu_i^2 e^{-(\lambda_j-\lambda_i)/\mu_i} - \mu_j^2 e^{-(\lambda_j-\lambda_i)/\mu_j}] & , \text{ if } \lambda_i < \lambda_j \,, \end{cases}$$

which in the limit $\mu_i \to \mu$, $\mu_j \to \mu$, reduces to

$$q_{ij} = \begin{cases} 1 - \frac{1}{2}\left[1 + \frac{\lambda_i-\lambda_j}{2\mu}\right] e^{-(\lambda_i-\lambda_j)/\mu} & , \text{ if } \lambda_i \geq \lambda_j \,, \\ \frac{1}{2}\left[1 + \frac{\lambda_j-\lambda_i}{2\mu}\right] e^{-(\lambda_j-\lambda_i)/\mu} & , \text{ if } \lambda_i < \lambda_j \,. \end{cases}$$

Let f be the $[0,+\infty] \to]0,1/2]$ mapping defined by

$$f(x) = \frac{1}{2}\left(1 + \frac{x}{2}\right) e^{-x} \,,$$

then, if $\lambda_i \geq \lambda_j \geq \lambda_k$, we obtain:

$$q_{ij} = 1 - f\left(\frac{\lambda_i - \lambda_j}{\mu}\right), \ q_{jk} = 1 - f\left(\frac{\lambda_j - \lambda_k}{\mu}\right), \ q_{ik} = 1 - f\left(\frac{\lambda_i - \lambda_k}{\mu}\right),$$

with $q_{ij} \geq 1/2$, $q_{jk} \geq 1/2$ and $q_{ik} \geq 1/2$. Since f is a one-to-one mapping, the generated probabilistic relation Q is isostochastic transitive w.r.t. the function h defined by

$$h(x, y) = 1 - f\left(f^{-1}(1-x) + f^{-1}(1-y)\right).$$

The associated strict t-conorm S_h is given by

$$S_h(x, y) = s^{-1}(s(x) + s(y))$$

with additive generator

$$s(x) = f^{-1}\left(\frac{1-x}{2}\right).$$

6.3.3 *Normally distributed dice (with same variance)*

We use the notation $\Phi(x)$ for the c.d.f. of the standard normal distribution $\mathrm{N}(0, 1)$ with expected value $\mu = 0$ and variance $\sigma^2 = 1$ (see Table 2). We will use the following well-known properties:

$$\Phi(-x) = 1 - \Phi(x), \ \ \Phi^{-1}(x) = -\Phi^{-1}(1-x). \tag{22}$$

Let $X_i \stackrel{d}{=} \mathrm{N}(\mu_i, \sigma_i^2)$, $X_j \stackrel{d}{=} \mathrm{N}(\mu_j, \sigma_j^2)$ and $X_k \stackrel{d}{=} \mathrm{N}(\mu_k, \sigma_k^2)$, then, since $X_j - X_i \stackrel{d}{=} \mathrm{N}(\mu_j - \mu_i, \sigma_i^2 + \sigma_j^2)$, we obtain

$$q_{ij} = \mathrm{Prob}\{X_i > X_j\} = \mathrm{Prob}\{X_j - X_i < 0\} = \Phi\left(\frac{\mu_i - \mu_j}{\sqrt{\sigma_i^2 + \sigma_j^2}}\right).$$

Now let all X_i have the same variance σ^2, and let us without loss of generality assume that $\mu_i \geq \mu_j \geq \mu_k$, then

$$q_{ij} = \Phi\left(\frac{\mu_i - \mu_j}{\sqrt{2\sigma^2}}\right), \ q_{jk} = \Phi\left(\frac{\mu_j - \mu_k}{\sqrt{2\sigma^2}}\right), \ q_{ik} = \Phi\left(\frac{\mu_i - \mu_k}{\sqrt{2\sigma^2}}\right),$$

and $q_{ij} \geq 1/2, q_{jk} \geq 1/2$ and $q_{ik} \geq 1/2$. Hence,

$$q_{ik} = \Phi\left(\Phi^{-1}(q_{ij}) + \Phi^{-1}(q_{jk})\right) ,$$

which proves that the probabilistic relation Q is isostochastic transitive w.r.t. the function h defined by

$$h(x, y) = \Phi\left(\Phi^{-1}(x) + \Phi^{-1}(y)\right) .$$

Note that due to (22) an alternative expression for the function h is

$$h(x, y) = 1 - \Phi\left(\Phi^{-1}(1 - x) + \Phi^{-1}(1 - y)\right) .$$

The associated strict t-conorm S_h is given by

$$S_h(x, y) = s^{-1}(s(x) + s(y))$$

with additive generator

$$s(x) = \Phi^{-1}\left(\frac{1 - x}{2}\right) .$$

An overview of the results obtained in the present section is presented in Table 2 where for the random variables with parametric distributions defined in Table 1, we list the function h w.r.t. which the probabilistic relation Q is isostochastic transitive.

In the cases of the unimodal uniform, Gumbel, Laplace and normal distributions we have fixed one of the two parameters in order to restrict the family to a one-parameter subfamily, mainly because with two free parameters, the formulae become utmost cumbersome. The one exception is the two-dimensional family of normal distributions for which, as we will see in the next section, a lot of simplifying steps in the computations allow to maintain the two parameters as free parameters.

Table 2
h-isostochastic transitivity for the dice models described in Table 1.

Name	Function h
Exponential Beta Pareto Gumbel	$\dfrac{x\,y}{x\,y+(1-x)(1-y)}$ associated to t-conorm $S_2^{\mathbf{H}}$ (also valid for discrete geometric dice)
Uniform	$1-\frac{1}{2}\left(\max(\sqrt{2(1-x)}+\sqrt{2(1-y)}-1,0)\right)^2$ associated to t-conorm $S_{1/2}^{\mathbf{SS}}$
Laplace	$1-f\left(f^{-1}(1-x)+f^{-1}(1-y)\right)$ with $f(x)=\frac{1}{2}\left(1+\frac{x}{2}\right)e^{-x}$
Normal	$\Phi\left(\Phi^{-1}(x)+\Phi^{-1}(y)\right)$ with $\Phi(x)=(\sqrt{2\pi})^{-1}\int_{-\infty}^{x}e^{-t^2/2}dt$

7 Normally distributed dice

Let us again consider a collection of normally distributed random variables $X_i \stackrel{d}{=} \mathrm{N}(\mu_i,\sigma_i^2)$. We know from the previous section that

$$q_{ij}=\Phi\left(\frac{\mu_i-\mu_j}{\sqrt{\sigma_i^2+\sigma_j^2}}\right),\ q_{jk}=\Phi\left(\frac{\mu_j-\mu_k}{\sqrt{\sigma_j^2+\sigma_k^2}}\right),\ q_{ik}=\Phi\left(\frac{\mu_i-\mu_k}{\sqrt{\sigma_i^2+\sigma_k^2}}\right).$$

Introducing the notation $\phi_{ij}=\sqrt{\sigma_i^2+\sigma_j^2}$, it follows from $\mu_i-\mu_k=(\mu_i-\mu_j)+(\mu_j-\mu_k)$, that

$$\phi_{ik}\Phi^{-1}(q_{ik})=\phi_{ij}\Phi^{-1}(q_{ij})+\phi_{jk}\Phi^{-1}(q_{jk})\,,$$

an equality which, since $\phi_{ik}=\phi_{ki}$, can be rewritten as

$$\phi_{ij}\Phi^{-1}(q_{ij})+\phi_{jk}\Phi^{-1}(q_{jk})+\phi_{ki}\Phi^{-1}(q_{ki})=0\,. \tag{23}$$

This formula turns out to be a key element in the proof of the following proposition.

Proposition 23 *The probabilistic relation generated by a collection of independent normal random variables is moderately stochastic transitive.*

Proof: Let us consider the case $q_{ij} \geq 1/2$ and $q_{jk} \geq 1/2$. It then follows that $\mu_i \geq \mu_j \geq \mu_k$, with as a consequence that also $q_{ik} \geq 1/2$. This means that $\gamma_{ijk} \geq \beta_{ijk} \geq 1/2$ and $\alpha_{ijk} = q_{ki}$. We have to prove that $1 - \alpha_{ijk} = q_{ik} \geq \min(\beta_{ijk}, \gamma_{ijk}) = \min(q_{ij}, q_{jk})$. Since Φ^{-1} is a strictly increasing function, this is equivalent to proving that the inequality $\Phi^{-1}(q_{ik}) \geq \min(\Phi^{-1}(q_{ij}), \Phi^{-1}(q_{jk}))$ is fulfilled. Using (23), we obtain that

$$\begin{aligned}\Phi^{-1}(q_{ik}) &= \frac{\phi_{ij}}{\phi_{ik}}\Phi^{-1}(q_{ij}) + \frac{\phi_{jk}}{\phi_{ik}}\Phi^{-1}(q_{jk}) \\ &\geq \frac{\phi_{ij}+\phi_{jk}}{\phi_{ik}} \min\left(\Phi^{-1}(q_{ij}), \Phi^{-1}(q_{jk})\right) .\end{aligned}$$

From the definition of ϕ_{ij}, it follows that $\phi_{ij} > 0$ and furthermore it can be shown that $|\phi_{jk}^2 - \phi_{ij}^2| \leq \phi_{ik}^2 \leq \phi_{ij}^2 + \phi_{jk}^2$, which implies that the numbers ϕ_{ik}^2, ϕ_{ij}^2, and ϕ_{jk}^2 are triangular numbers, since they satisfy the classical triangular conditions. From the rightmost inequality of this double inequality, we derive that

$$\phi_{ij} + \phi_{jk} = \sqrt{\phi_{ij}^2 + \phi_{jk}^2 + 2\phi_{ij}\phi_{jk}} \geq \sqrt{\phi_{ij}^2 + \phi_{jk}^2} \geq \phi_{ik} ,$$

which completes the proof. □

8 Conclusion

In this paper, we have introduced generalized dice models for comparing pairwise independent random variables with arbitrary discrete or continuous distributions. The probabilistic relation generated by the random variables can be seen as a graded generalization of the concept of stochastic dominance. It is well known that, in general, probabilistic relations can show cyclic behaviour and are therefore not transitive. The framework of cycle-transitivity is well suited for investigating such relations, since cycle-transitivity does not exclude cyclic behaviour. In our study of the transitivity properties of generalized dice models, we have highlighted the special role of dice-transitivity and self-dual upper bound functions. Finally, we have investigated the transitivity of some specific dice models and have laid bare interesting links with t-conorms and stochastic transitivity. The transitivity properties of generalized dice models with dependent random variables will be investigated in upcoming work.

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