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Cyclic Evaluation of Transitivity of Reciprocal Relations

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Abstract A general framework for studying the transitivity of reciprocal relations is presented. The key feature is the cyclic evaluation of transitivity: triangles (i.e. any three points) are visited in a cyclic manner. An upper bound function acting upon the ordered weights encountered provides an upper bound for the 'sum minus 1' of these weights. Commutative quasi-copulas allow to translate a general definition of fuzzy transitivity (when applied to reciprocal relations) elegantly into the framework of cycle-transitivity. Similarly, a general notion of stochastic transitivity corresponds to a particular class of upper bound functions. Ample attention is given to self-dual upper bound functions.

Key words Copulas – Reciprocal relations – Stochastic transitivity – T-norms – Transitivity – T-transitivity.

1 Introduction

The goal of this paper is to develop a general framework for studying various kinds of transitivity of reciprocal relations, i.e. $[0, 1]$-valued relations Q satisfying $Q(a, b) + Q(b, a) = 1$. Such relations are known under various names such as ipsodual relations or probabilistic relations [11]. A related concept is that of a comparison function, which takes values in $\mathbb{R}$ instead of $[0, 1]$ and satisfies the condition $g(x, y) = -g(y, x)$ [12].

Comparison functions and reciprocal relations are a convenient tool for expressing the result of the pairwise comparison of a set of alternatives [8] and appear in various fields such as game theory [12], voting theory [17, 26] and psychological studies on preference and discrimination in (individual or collective) decision-making methods [11]. Reciprocal relations are particularly popular in fuzzy set theory where they are used for representing intensities of preference [5, 20, 31].

In group decision making, reciprocal relations represent collective preferences and are built from individual preferences, either by aggregation methods [16] or consensus-reaching processes [20]. In social choice theory, there is a vast literature on the study of choice rules [6, 19, 26] (resp. choice correspondences [12, 23]) given preferences expressed in terms of reciprocal relations (resp. comparison functions).

Whatever relational representation is employed for intensities of preference, transitivity is always an interesting, often desirable property. In the context of fuzzy preference modelling, for instance, T-transitivity of fuzzy (i.e. $[0,1]$-valued) relations is an indispensable notion [3, 7, 14, 28]. Some types of transitivity have been devised specifically for reciprocal relations, such as various types of stochastic transitivity [13, 24, 27]. We start out this paper by recalling some of these notions in Section 2.

Although T-transitivity has been devised for fuzzy relations, and does not a priori make sense for reciprocal relations, we begin Section 3 with a careful study of $T_{\mathbf{P}}$-transitivity for reciprocal relations. Our observations will motivate the introduction of a new type of transitivity, which is essentially characterized by its cyclic evaluation, whence the term cycle-transitivity. The central concept is that of an upper bound function. Particular attention is paid to self-dual upper bound functions.

In Section 4, we show how fuzzy transitivity, and in particular T-transitivity, fits into the new framework. Commutative quasi-copulas, and in particular members of the Frank t-norm family, permit an elegant reformulation. In Section 5, we propose a broad definition of stochastic transitivity, of which strong, moderate and weak stochastic transitivity are well-known instances. It is shown under which conditions this type of transitivity can be cast into the cycle-transitivity framework as well. Finally, the discussion of self-duality leads to remarkable results, attributing a particular role to t-conorms.

2 Transitivity of fuzzy and reciprocal relations

2.1 Transitivity of fuzzy relations

Transitivity is a simple, yet powerful property of relations. A (binary) relation R on a universe A (often referred to as the *set of alternatives*) is called *transitive* if for any $(a,b,c) \in A^3$ it holds that

$$((a,b) \in R \wedge (b,c) \in R) \Rightarrow (a,c) \in R\,. \tag{1}$$

Identifying a relation with its characteristic mapping, i.e. defining

$$R(a,b) = \begin{cases} 1 & \text{, if } (a,b) \in R\,, \\ 0 & \text{, if } (a,b) \notin R\,, \end{cases}$$

transitivity can be stated equivalently as

$$(R(a,b) = 1 \wedge R(b,c) = 1) \Rightarrow R(a,c) = 1\,.$$

However, many other equivalent formulations may be devised, such as

$$(R(a,b) \geq \alpha \wedge R(b,c) \geq \alpha) \Rightarrow R(a,c) \geq \alpha\,, \tag{2}$$

for any $\alpha > 0$. Alternatively, transitivity can also be expressed in the following functional form

$$\min(R(a,b), R(b,c)) \leq R(a,c)\,. \tag{3}$$

Note that on $\{0,1\}^2$ the minimum operation is nothing else but the Boolean conjunction.

In the setting of fuzzy set theory, formulation (3) has led to the popular notion of T-transitivity, where a t-norm T is used as a generalization of the Boolean conjunction. A *fuzzy relation* R on A is an $A^2 \to [0,1]$ mapping that expresses the degree of relationship between elements of A: $R(a,b) = 0$ means a and b are not related at all, $R(a,b) = 1$ expresses full relationship, while $R(a,b) \in]0,1[$ indicates a partial degree of relationship only.

Definition 1 [29] *A binary operation* $T : [0,1]^2 \to [0,1]$ *is called a t-norm if it satisfies:*

(i) Neutral element 1*:* $(\forall x \in [0,1])(T(x,1) = T(1,x) = x)$.
(ii) Monotonicity: T is increasing in each variable.
(iii) Commutativity: $(\forall (x,y) \in [0,1]^2)(T(x,y) = T(y,x))$.
(iv) Associativity: $(\forall (x,y,z) \in [0,1]^3)(T(x,T(y,z)) = T(T(x,y),z))$.

To any t-norm T corresponds a dual t-conorm S defined by

$$S(x,y) = 1 - T(1-x, 1-y)\,. \tag{4}$$

More formally, a t-conorm is a binary operation on $[0,1]$ which satisfies properties (ii)–(iv) above and has as neutral element 0. For a recent monograph on t-norms and t-conorms, we refer to [21].

Definition 2 *Let T be a t-norm. A fuzzy relation R on A is called T-transitive if for any* $(a,b,c) \in A^3$ *it holds that*

$$T(R(a,b), R(b,c)) \leq R(a,c)\,. \tag{5}$$

The three main continuous t-norms are the minimum operator $T_{\mathbf{M}}$, the algebraic product $T_{\mathbf{P}}$ and the Łukasiewicz t-norm $T_{\mathbf{L}}$ (defined by $T_{\mathbf{L}}(x,y) = \max(x+y-1, 0)$). The smallest t-norm is the drastic product $T_{\mathbf{D}}$, which is right-continuous only and is 0 everywhere up to the boundary condition $T_{\mathbf{D}}(x,1) = T_{\mathbf{D}}(1,x) = x$.

2.2 Transitivity of reciprocal relations

Another class of $A^2 \to [0,1]$ mappings are the so-called *reciprocal relations* Q satisfying

$$Q(a,b) + Q(b,a) = 1\,, \tag{6}$$

for any $a, b \in A$. For such relations, it holds in particular that $Q(a,a) = 1/2$. Transitivity properties for reciprocal relations rather have the logical flavour of expression (2). There exist various kinds of stochastic transitivity for reciprocal relations [8,24]. For instance, a reciprocal relation Q on A is called *weakly stochastic transitive* if for any $(a,b,c) \in A^3$ it holds that

$$(Q(a,b) \geq 1/2 \ \wedge \ Q(b,c) \geq 1/2) \ \Rightarrow \ Q(a,c) \geq 1/2\,, \tag{7}$$

which corresponds to the choice of $\alpha = 1/2$ in (2).

Next, let R be a complete ($\{0,1\}$-valued) relation on A, which means that $\max(R(a,b), R(b,a)) = 1$ for any $a, b \in A$. Then R has an equivalent $\{0, 1/2, 1\}$-valued reciprocal representation Q given by

$$Q(a,b) = \begin{cases} 1 & \text{, if } R(a,b) = 1 \text{ and } R(b,a) = 0\,, \\ 1/2 & \text{, if } R(a,b) = R(b,a) = 1\,, \\ 0 & \text{, if } R(a,b) = 0 \text{ and } R(b,a) = 1\,. \end{cases}$$

Or in a more compact arithmetic form:

$$Q(a,b) = \frac{1 + R(a,b) - R(b,a)}{2}\,. \tag{8}$$

One easily verifies that R is transitive if and only if its reciprocal representation Q satisfies, for any $(a,b,c) \in A^3$:

$$(Q(a,b) \geq 1/2 \ \wedge \ Q(b,c) \geq 1/2) \ \Rightarrow \ Q(a,c) = \max(Q(a,b), Q(b,c))\,. \tag{9}$$

Similarly, a *weakly complete* fuzzy relation R, i.e. one satisfying

$$R(a,b) + R(b,a) \geq 1\,,$$

for any $a, b \in A$, can be transformed into a (non-equivalent, yet interesting) reciprocal representation $Q = P + I/2$, with P and I the (fuzzy) strict preference and indifference components of the *fuzzy preference structure* (P, I, J) generated from R by means of $T_{\mathbf{L}}$ [10,32]:

$$\begin{aligned} P(a,b) &= T_{\mathbf{M}}(R(a,b), 1 - R(b,a)) = 1 - R(b,a)\,, \\ I(a,b) &= T_{\mathbf{L}}(R(a,b), R(b,a)) = R(a,b) + R(b,a) - 1\,, \\ J(a,b) &= T_{\mathbf{L}}(1 - R(a,b), 1 - R(b,a)) = 0\,. \end{aligned}$$

Note that the corresponding expression for Q is formally the same as (8). For an introduction to fuzzy preference structures, we refer to [9].

3 Cycle-transitivity

3.1 Notations

Consider an arbitrary universe A. For a reciprocal relation Q on A, we write $q_{ab} := Q(a,b)$. For any $(a,b,c) \in A^3$, let

$$\begin{aligned} \alpha_{abc} &= \min(q_{ab}, q_{bc}, q_{ca})\,, \\ \beta_{abc} &= \text{median}(q_{ab}, q_{bc}, q_{ca})\,, \\ \gamma_{abc} &= \max(q_{ab}, q_{bc}, q_{ca})\,. \end{aligned} \tag{10}$$

It then obviously holds that

$$\alpha_{abc} \le \beta_{abc} \le \gamma_{abc}\,, \tag{11}$$

and also

$$\alpha_{abc} = \alpha_{bca} = \alpha_{cab}\,, \quad \beta_{abc} = \beta_{bca} = \beta_{cab}\,, \quad \gamma_{abc} = \gamma_{bca} = \gamma_{cab}\,. \tag{12}$$

On the other hand, the reciprocity of Q implies that

$$\alpha_{cba} = 1 - \gamma_{abc}\,, \quad \beta_{cba} = 1 - \beta_{abc}\,, \quad \gamma_{cba} = 1 - \alpha_{abc}\,. \tag{13}$$

3.2 Product-transitivity revisited

To point out a possible way of generalizing T-transitivity (for reciprocal relations), we consider $T_{\mathbf{P}}$-transitivity for a reciprocal relation Q on A. For any $a,b,c \in A$, there are six conditions to be satisfied, namely

$$\begin{aligned} q_{ac}\,q_{cb} \,&\le\, q_{ab}\,, \quad & q_{ba}\,q_{ac} &\le q_{bc}\,, \quad & q_{cb}\,q_{ba} &\le q_{ca}\,, \\ q_{bc}\,q_{ca} \,&\le\, q_{ba}\,, \quad & q_{ca}\,q_{ab} &\le q_{cb}\,, \quad & q_{ab}\,q_{bc} &\le q_{ac}\,. \end{aligned}$$

Since Q is reciprocal, these conditions can be expressed in terms of α_{abc}, β_{abc} and γ_{abc} solely, as follows

$$\begin{aligned} (1-\beta_{abc})(1-\gamma_{abc}) \,&\le\, \alpha_{abc}\,, \\ (1-\alpha_{abc})(1-\gamma_{abc}) \,&\le\, \beta_{abc}\,, \\ (1-\alpha_{abc})(1-\beta_{abc}) \,&\le\, \gamma_{abc}\,, \\ \beta_{abc}\,\gamma_{abc} \,&\le\, 1-\alpha_{abc}\,, \\ \alpha_{abc}\,\gamma_{abc} \,&\le\, 1-\beta_{abc}\,, \\ \alpha_{abc}\,\beta_{abc} \,&\le\, 1-\gamma_{abc}\,. \end{aligned} \tag{14}$$

The first three inequalities of (14) can be rewritten as

$$\begin{aligned} \beta_{abc}\gamma_{abc} \,&\le\, \alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1\,, \\ \alpha_{abc}\gamma_{abc} \,&\le\, \alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1\,, \\ \alpha_{abc}\beta_{abc} \,&\le\, \alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1\,. \end{aligned}$$

From (11) it follows that $\alpha_{abc}\,\beta_{abc} \leq \alpha_{abc}\,\gamma_{abc} \leq \beta_{abc}\,\gamma_{abc}$. Therefore only the first inequality should be withheld as a condition for $T_{\mathbf{P}}$-transitivity. Similarly, the last three inequalities of (14) can be rewritten as

$$\begin{aligned}
\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 &\leq 1 - (1-\beta_{abc})(1-\gamma_{abc})\,,\\
\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 &\leq 1 - (1-\alpha_{abc})(1-\gamma_{abc})\,,\\
\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 &\leq 1 - (1-\alpha_{abc})(1-\beta_{abc})\,.
\end{aligned}$$

From (11) it now follows that only the last inequality should be retained. The six inequalities (14) are therefore equivalent to the double inequality

$$\beta_{abc}\,\gamma_{abc} \leq \alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \leq 1 - (1-\alpha_{abc})(1-\beta_{abc})\,. \tag{15}$$

The way we arrived at this double inequality immediately shows that if it holds for $(a,b,c) \in A^3$, then it also holds for all permutations of (a,b,c). A direct proof of this claim, however, provides us with some further insights. Let us denote the upper and lower bounds in (15) as $u(\alpha_{abc},\beta_{abc})$ and $l(\beta_{abc},\gamma_{abc})$, respectively. We observe the following type of duality

$$l(\beta_{abc},\gamma_{abc}) = 1 - u(1-\gamma_{abc}, 1-\beta_{abc})\,. \tag{16}$$

Suppose (15) holds for (a,b,c), then (13) and (16) lead to

$$\begin{aligned}
\alpha_{cba} + \beta_{cba} + \gamma_{cba} - 1 &= 1 - (\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1)\\
&\geq 1 - u(\alpha_{abc},\beta_{abc})\\
&= 1 - u(1-\gamma_{cba}, 1-\beta_{cba}) = l(\beta_{cba},\gamma_{cba})\,.
\end{aligned}$$

Similarly, we obtain

$$\alpha_{cba} + \beta_{cba} + \gamma_{cba} - 1 \leq u(\alpha_{cba},\beta_{cba})\,.$$

Hence, (15) also holds for (c,b,a).

3.3 Definition of cycle-transitivity

The simple formulation (15)–(16) of $T_{\mathbf{P}}$-transitivity for reciprocal relations has been our source of inspiration for a new type of transitivity for reciprocal relations. Let us denote $\Delta = \{(x,y,z) \in [0,1]^3 \mid x \leq y \leq z\}$ and consider a function $U : \Delta \to \mathbb{R}$, then, in analogy to (15), we could call a reciprocal relation Q on A transitive w.r.t. U if for any $a,b,c \in A$ it holds that

$$1-U(1-\gamma_{abc}, 1-\beta_{abc}, 1-\alpha_{abc}) \leq \alpha_{abc}+\beta_{abc}+\gamma_{abc}-1 \leq U(\alpha_{abc},\beta_{abc},\gamma_{abc})\,.$$

In case of $T_{\mathbf{P}}$-transitivity, the corresponding function $U_{\mathbf{P}}$ is given by

$$U_{\mathbf{P}}(\alpha,\beta,\gamma) = 1 - (1-\alpha)(1-\beta) = \alpha + \beta - \alpha\beta\,. \tag{17}$$

The minimal requirement we impose is that the reciprocal representation Q of any transitive complete relation R given in (8) has this type of transitivity. To that end, U should satisfy the following conditions:

$$\begin{array}{cc} U(0,1/2,1) \geq 1/2\,, & U(1/2,1/2,1/2) \geq 1/2\,, \\ U(0,0,1) \geq 0\,, & U(0,1,1) \geq 1\,. \end{array} \tag{18}$$

These conditions are for instance satisfied for any $U \geq$ median.

Similarly, a kind of maximal requirement could be to insist that the only $\{0,1/2,1\}$-valued reciprocal relations that are transitive w.r.t. U are the reciprocal representations of transitive complete relations. To that end, U should satisfy the following additional conditions

$$\begin{array}{ccc} U(0,0,0) < -1 & \text{or} & U(1,1,1) < 2\,, \\ U(0,0,1/2) < -1/2 & \text{or} & U(1/2,1,1) < 3/2\,, \\ U(0,1/2,1/2) < 0 & \text{or} & U(1/2,1/2,1) < 1\,. \end{array} \tag{19}$$

As this requirement would seriously limit the generality of our framework, it will not be included in our basic definitions. However, it will be commented upon where relevant.

The proposed double inequality actually restricts the possible values of α_{abc}, β_{abc} and γ_{abc} in two consecutive steps. First, the lower bound should not exceed the upper bound, and, second, if this is indeed not the case, then the value $\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1$ should be located between these bounds. In order not to exclude any $(\alpha_{abc}, \beta_{abc}, \gamma_{abc})$ a priori, we propose the following definitions.

Definition 3 *A function $U : \Delta \to \mathbb{R}$ is called an upper bound function if it satisfies:*

(i) $U(0,0,1) \geq 0$ and $U(0,1,1) \geq 1$;
(ii) for any $(\alpha,\beta,\gamma) \in \Delta$:

$$U(\alpha,\beta,\gamma) + U(1-\gamma,1-\beta,1-\alpha) \geq 1\,. \tag{20}$$

The class of upper bound functions is denoted $\mathcal{U}$.

Note that the definition of an upper bound function does not include any monotonicity condition. The function $L : \Delta \to \mathbb{R}$ defined by

$$L(\alpha,\beta,\gamma) = 1 - U(1-\gamma,1-\beta,1-\alpha) \tag{21}$$

is called the *dual lower bound function* of a given upper bound function U. Inequality (20) then simply expresses that $L \leq U$. Note that the conditions $U(0,1/2,1) \geq 1/2$ and $U(1/2,1/2,1/2) \geq 1/2$ have been omitted from (18) as they follow from (20). One easily verifies that $U_{\mathbf{P}}$ belongs to $\mathcal{U}$ and satisfies (19).

Definition 4 *A reciprocal relation Q on A is called cycle-transitive w.r.t. an upper bound function U if for any $(a,b,c) \in A^3$ it holds that*

$$L(\alpha_{abc}, \beta_{abc}, \gamma_{abc}) \leq \alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \leq U(\alpha_{abc}, \beta_{abc}, \gamma_{abc}), \tag{22}$$

where L is the dual lower bound function of U.

Using the above terminology, the results of Subsection 3.2 can be rephrased as follows: a reciprocal relation Q is $T_{\mathbf{P}}$-transitive if and only if it is cycle-transitive w.r.t. $U_{\mathbf{P}}$. Due to the built-in duality, it still holds that if (22) is true for some (a,b,c), then this is also the case for any permutation of (a,b,c). In practice, it is therefore sufficient to check (22) for a single permutation of any $(a,b,c) \in A^3$. Alternatively, due to the same duality, it is also sufficient to verify the right-hand inequality (or equivalently, the left-hand inequality) for two permutations of any $(a,b,c) \in A^3$ (not being cyclic permutations of one another), e.g. (a,b,c) and (c,b,a).

Proposition 1 *A reciprocal relation Q on A is cycle-transitive w.r.t. an upper bound function U if for any $(a,b,c) \in A^3$ it holds that*

$$\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \leq U(\alpha_{abc}, \beta_{abc}, \gamma_{abc}). \tag{23}$$

Note that a value of $U(\alpha,\beta,\gamma)$ equal to 2 will often be used to express that for the given values there is no restriction at all (indeed, $\alpha+\beta+\gamma-1$ is always bounded by 2). For two upper bound functions such that $U_1 \leq U_2$, it clearly holds that cycle-transitivity w.r.t. U_1 implies cycle-transitivity w.r.t. U_2. It is clear that $U_1 \leq U_2$ is not a necessary condition for the latter implication to hold.

Two upper bound functions U_1 and U_2 will be called *equivalent* if for any $(\alpha,\beta,\gamma) \in \Delta$ it holds that

$$\alpha+\beta+\gamma-1 \leq U_1(\alpha,\beta,\gamma)$$

is equivalent to

$$\alpha+\beta+\gamma-1 \leq U_2(\alpha,\beta,\gamma).$$

Suppose, for instance, that the inequality $\alpha+\beta+\gamma-1 \leq U_1(\alpha,\beta,\gamma)$ can be rewritten as

$$\alpha \leq h(\beta,\gamma),$$

then an equivalent upper bound function U_2 is given by

$$U_2(\alpha,\beta,\gamma) = \beta+\gamma-1+h(\beta,\gamma).$$

In this way, it is often possible to find an equivalent upper bound function in only two of the variables α, β and γ.

A more general way of obtaining equivalent upper bound functions is the following. For any $\mu > 0$, the inequality

$$\alpha+\beta+\gamma-1 \leq U(\alpha,\beta,\gamma)$$

is clearly equivalent to

$$\alpha+\beta+\gamma-1 \leq \frac{U(\alpha,\beta,\gamma)-(1-\mu)(\alpha+\beta+\gamma-1)}{\mu}\,.$$

Hence, cycle-transitivity w.r.t. U is equivalent to cycle-transitivity w.r.t. U_μ defined by

$$U_\mu(\alpha,\beta,\gamma) = \frac{U(\alpha,\beta,\gamma)-(1-\mu)(\alpha+\beta+\gamma-1)}{\mu}\,. \tag{24}$$

One easily verifies that $U_\mu \in \mathcal{U}$. Note that also the additional conditions (19) are preserved under the above transformation.

3.4 Self-dual upper bound functions

If it happens that in (20) the equality holds for all $(\alpha,\beta,\gamma) \in \Delta$, i.e.

$$U(\alpha,\beta,\gamma) + U(1-\gamma,1-\beta,1-\alpha) = 1\,, \tag{25}$$

then the upper bound function U is said to be *self-dual*, since in that case it coincides with its dual lower bound function L. Consequently, then also (22) and (23) can only hold with equality. Furthermore, it holds that $U(0,0,1) = 0$ and $U(0,1,1) = 1$. Note that if U is self-dual, then also any upper bound function U_μ defined in (24) is self-dual.

The simplest example of a self-dual upper bound function is the median, i.e. $U_{\mathbf{M}}(\alpha,\beta,\gamma) = \beta$, and further on we will prove that this is precisely the upper bound function corresponding to $T_{\mathbf{M}}$-transitivity of reciprocal relations, when reformulated in the framework of cycle-transitivity.

Another example of a self-dual upper bound function is the function U_E defined by

$$U_E(\alpha,\beta,\gamma) = \alpha\beta + \alpha\gamma + \beta\gamma - 2\alpha\beta\gamma\,. \tag{26}$$

Cycle-transitivity w.r.t. U_E of a reciprocal relation Q on A can also be expressed as

$$\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 = \alpha_{abc}\beta_{abc} + \alpha_{abc}\gamma_{abc} + \beta_{abc}\gamma_{abc} - 2\alpha_{abc}\beta_{abc}\gamma_{abc}\,,$$

or, equivalently, as:

$$\alpha_{ijk}\beta_{ijk}\gamma_{ijk} = (1-\alpha_{ijk})(1-\beta_{ijk})(1-\gamma_{ijk})\,.$$

It is then easy to see that cycle-transitivity w.r.t. U_E is equivalent to the concept of multiplicative transitivity [31]. Recall that a reciprocal relation Q on A is called *multiplicatively transitive* if for any $(a,b,c) \in A^3$ it holds that

$$\frac{Q(a,c)}{Q(c,a)} = \frac{Q(a,b)}{Q(b,a)} \cdot \frac{Q(b,c)}{Q(c,b)}\,.$$

Note that the cycle-transitive version is more appropriate as it avoids division by zero.

It is not difficult to characterize the subfamily of self-dual upper bound functions that are polynomials. Indeed, introducing the new variables $\alpha' = \alpha - 1/2$, $\beta' = \beta - 1/2$, $\gamma' = \gamma - 1/2$, and the new function U' defined by

$$U'(\alpha', \beta', \gamma') = U(\alpha' + 1/2, \beta' + 1/2, \gamma' + 1/2) - 1/2\,,$$

the self-duality of U becomes equivalent to

$$U'(\alpha', \beta', \gamma') = -U'(-\gamma', -\beta', -\alpha')\,,$$

which should hold for all $-1/2 \leq \alpha' \leq \beta' \leq \gamma' \leq 1/2$. Using standard algebraic manipulations we can show that the polynomial solutions of the latter functional equation are given by

$$U'(\alpha', \beta', \gamma') = \sum_{i,j,k=0}^{+\infty} c_{ijk} (\alpha'\gamma')^i \beta'^j \left[\alpha'^k + (-1)^{j+k+1} \gamma'^k\right]\,,$$

where the c_{ijk} are arbitrary (real) coefficients. Note that the latter expression not only defines self-dual polynomials, but also self-dual analytic functions (in other words, the (triple) series converges for all $-1/2 \leq \alpha' \leq \beta' \leq \gamma' \leq 1/2$). Returning to the original variables, the polynomial (or analytic) self-dual upper bound functions U are given by

$$\begin{aligned} U(\alpha, \beta, \gamma) = \frac{1}{2} + \sum_{i,j,k=0}^{+\infty} & c_{ijk} (\alpha - 1/2)^i \, (\beta - 1/2)^j \, (\gamma - 1/2)^i \\ & \times \left[(\alpha - 1/2)^k + (-1)^{j+k+1} (\gamma - 1/2)^k\right]\,, \end{aligned} \tag{27}$$

where the coefficients c_{ijk} are further restricted in order to ensure the conditions $U(0, 0, 1) = 0$ and $U(0, 1, 1) = 1$.

By setting $c_{010} = 1/2$ and all other c_{ijk} to zero in (27), the self-dual upper bound function $U_{\mathbf{M}}$ is retrieved, while choosing $c_{110} = -2$, $c_{001} = c_{010} = 1/2$ and all other $c_{ijk} = 0$, leads to the self-dual upper bound function U_E.

4 Fuzzy transitivity as cycle-transitivity

4.1 Fuzzy transitivity

In this section, we reconsider the notion of T-transitivity. Instead of t-norms, we consider the more general class of conjunctors.

Definition 5 *A binary operation $f : [0, 1]^2 \to [0, 1]$ is called a conjunctor if it has the following properties:*

(i) Its restriction to $\{0, 1\}^2$ coincides with the Boolean conjunction.
(ii) Monotonicity: f is increasing in each variable.

The following definition generalizes Definition 2.

Definition 6 *Let f be a conjunctor. A fuzzy relation R on A is called f-transitive if for any $(a,b,c) \in A^3$ it holds that*

$$f(R(a,b), R(b,c)) \leq R(a,c)\,. \tag{28}$$

Typical examples of conjunctors are binary operations on $[0,1]$ that satisfy (ii) and have 1 as neutral element 1, i.e. $f(x,1) = f(1,x) = x$ for any $x \in [0,1]$. Such conjunctors are bounded from above by $T_{\mathbf{M}}$, i.e. $f(x,y) \leq \min(x,y)$, and have 0 as absorbing element, i.e. $f(x,0) = f(0,x) = 0$, for any $x \in [0,1]$.

In this paper, we are mainly interested in two particular classes of conjunctors with neutral element 1: the class of t-norms mentioned in Subsection 2.1, and the class of (quasi-)copulas, which just as t-norms, finds its origin in the study of probabilistic metric spaces [29]. Where t-norms have the additional properties of commutativity and associativity, quasi-copulas have the 1-Lipschitz property, while copulas have the property of moderate growth.

Definition 7 [1,18,25] *A binary operation $C : [0,1]^2 \to [0,1]$ is called a quasi-copula if it satisfies:*

(i) Neutral element 1*:* $(\forall x \in [0,1])(C(x,1) = C(1,x) = x)\,.$
(i') Absorbing element 0*:* $(\forall x \in [0,1])(C(x,0) = C(0,x) = 0)\,.$
(ii) Monotonicity: C is increasing in each variable.
(iii) 1-Lipschitz property: $(\forall (x_1, x_2, y_1, y_2) \in [0,1]^4)$

$$(|C(x_1,y_1) - C(x_2,y_2)| \leq |x_1 - x_2| + |y_1 - y_2|)\,.$$

If instead of (iii), C satisfies

(iv) Moderate growth: $(\forall (x_1, x_2, y_1, y_2) \in [0,1]^4)$

$$((x_1 \leq x_2 \,\wedge\, y_1 \leq y_2) \Rightarrow C(x_1,y_1) + C(x_2,y_2) \geq C(x_1,y_2) + C(x_2,y_1))\,,$$

then it is called a copula.

Note that in case of a quasi-copula condition (i') is superfluous. For a copula, condition (ii) can be omitted (as it follows from (iv) and (i')). As implied by the terminology used, any copula is a quasi-copula, and therefore has the 1-Lipschitz property; the opposite is, of course, not true. It is well known that a copula is a t-norm if and only if it is associative; conversely, a t-norm is a copula if and only if it is 1-Lipschitz. Finally, note that for any quasi-copula C it holds that $T_{\mathbf{L}} \leq C \leq T_{\mathbf{M}}$.

4.2 Fuzzy transitivity as cycle-transitivity

A first immediate observation is the following proposition. Although reciprocal relations do not have a fuzzy interpretation, we may attempt to study their f-transitivity. Indeed, f-transitivity of reciprocal relations is not a void notion, as for instance the reciprocal relation Q on $\{a,b,c\}$ defined by $Q(a,b) = Q(b,c) = Q(a,c) = 1$ is f-transitive for any conjunctor f.

Proposition 2 *Let f be a commutative conjunctor such that $f \leq T_{\mathbf{M}}$. A reciprocal relation Q on A is f-transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_f defined by*

$$U_f(\alpha,\beta,\gamma) = \min(\alpha+\beta-f(\alpha,\beta), \beta+\gamma-f(\beta,\gamma), \gamma+\alpha-f(\gamma,\alpha))\,. \quad (29)$$

Proof First of all, a tedious, yet simple verification shows that for any conjunctor f the property $f \leq T_{\mathbf{M}}$ guarantees that the function U_f defined in (29) belongs to $\mathcal{U}$.

Consider a reciprocal relation Q on A and $(a,b,c) \in A^3$. Assume e.g. that $q_{ab} = \alpha_{abc}$, $q_{bc} = \beta_{abc}$ and $q_{ca} = \gamma_{abc}$. The six inequalities of type (28), guaranteeing f-transitivity, can be brought, by adding appropriate terms to both sides of the inequalities, into the following form (also omitting the indices abc):

$$\begin{aligned}
f(1-\gamma,1-\beta)+\beta+\gamma-1 &\leq \alpha+\beta+\gamma-1\,,\\
f(1-\alpha,1-\gamma)+\alpha+\gamma-1 &\leq \alpha+\beta+\gamma-1\,,\\
f(1-\beta,1-\alpha)+\alpha+\beta-1 &\leq \alpha+\beta+\gamma-1\,,\\
\alpha+\beta+\gamma-1 &\leq -f(\beta,\gamma)+\beta+\gamma\,,\\
\alpha+\beta+\gamma-1 &\leq -f(\gamma,\alpha)+\alpha+\gamma\,,\\
\alpha+\beta+\gamma-1 &\leq -f(\alpha,\beta)+\alpha+\beta\,.
\end{aligned}$$

Similarly as for $T_{\mathbf{P}}$-transitivity, these six inequalities are equivalent to the double inequality

$$L_f(\alpha,\beta,\gamma) \leq \alpha+\beta+\gamma-1 \leq U_f(\alpha,\beta,\gamma)\,,$$

with U_f given by (29) and L_f the dual lower bound function defined by (21). Due to the commutativity of f, any other case, such as $q_{ab} = \alpha_{abc}$, $q_{bc} = \gamma_{abc}$ and $q_{ca} = \beta_{abc}$, leads to the same result. □

Note that in general the additional conditions (19) are not satisfied by an upper bound function of type (29). This is only the case when $f(1/2,1/2) > 0$, a condition that is e.g. not fulfilled for $f = T_{\mathbf{L}}$.

4.3 The case of commutative quasi-copulas and copulas

Proposition 2 does not sufficiently emphasize the relevance of the concept of cycle-transitivity. It would be interesting to establish sufficient conditions bringing the upper bound function U_f in a simpler form, in analogy to the result obtained for $T_{\mathbf{P}}$.

Proposition 3 *Let f be a commutative conjunctor such that $f \leq T_{\mathbf{M}}$. If f is 1-Lipschitz, then a reciprocal relation Q on A is f-transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_f defined by*

$$U_f(\alpha,\beta,\gamma) = \alpha+\beta-f(\alpha,\beta)\,. \quad (30)$$

Proof First, we observe that due to the monotonicity and commutativity of f, the 1-Lipschitz property of f can be stated equivalently as

$$y - f(x,y) \leq z - f(x,z)\,, \tag{31}$$

for any x and any $y \leq z$.

In view of Proposition 2, it is sufficient to show that

$$\min(\alpha+\beta-f(\alpha,\beta), \beta+\gamma-f(\beta,\gamma), \gamma+\alpha-f(\gamma,\alpha)) = \alpha+\beta-f(\alpha,\beta)\,,$$

for any $(\alpha,\beta,\gamma) \in \Delta$. As a double application of (31) leads to

$$\beta - f(\alpha,\beta) \leq \gamma - f(\gamma,\alpha)$$

and

$$\alpha - f(\alpha,\beta) \leq \gamma - f(\beta,\gamma)\,,$$

the proposition holds indeed. □

Let us characterize the upper bound functions U_f of the form (30) that are self-dual upper bound functions.

Proposition 4 *The minimum operator $T_{\mathbf{M}}$ is the only 1-Lipschitz commutative conjunctor $f \leq T_{\mathbf{M}}$ such that the associated upper bound function U_f is self-dual.*

Proof The self-dual upper bound functions of the form (30) are characterized by the equality

$$\alpha+\beta-f(\alpha,\beta)+1-\gamma+1-\beta-f(1-\gamma,1-\beta) = 1\,,$$

for any $(\alpha,\beta,\gamma) \in \Delta$. Rewriting this equality in the form

$$f(\alpha,\beta)+f(1-\gamma,1-\beta) = \alpha+(1-\gamma)\,,$$

and taking into account that $f \leq T_{\mathbf{M}}$, the only function f that identically satisfies this equality is $f = T_{\mathbf{M}}$. □

Note that the corresponding (self-dual) upper bound function is then simply given by $U_{\mathbf{M}}(\alpha,\beta,\gamma) = \alpha+\beta-\min(\alpha,\beta) = \beta$, as announced earlier. If we replace the condition $f \leq T_{\mathbf{M}}$ in Proposition 3 by the stronger condition (in the given context) that f should have as neutral element 1, then we are in fact dealing with a commutative quasi-copula.

Corollary 1 *Let C be a commutative quasi-copula. A reciprocal relation Q on A is C-transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_C defined by*

$$U_C(\alpha,\beta,\gamma) = \alpha+\beta-C(\alpha,\beta)\,. \tag{32}$$

In case of a copula, the operation in (32) is known as the *dual of the copula* [25].

Corollary 2 *Let C be a commutative copula. A reciprocal relation Q on A is C-transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_C defined by*

$$U_C(\alpha, \beta, \gamma) = \tilde{C}(\alpha, \beta)\,, \tag{33}$$

where

$$\tilde{C}(\alpha, \beta) = \alpha + \beta - C(\alpha, \beta) \tag{34}$$

is the dual of the copula C.

Note that besides the dual of a copula, one also defines the *co-copula* C^* of a copula C by

$$C^*(x, y) = 1 - C(1 - x, 1 - y)\,, \tag{35}$$

and the *survival copula* $\hat{C}$ associated to the copula C by

$$\hat{C}(x, y) = x + y - 1 + C(1 - x, 1 - y)\,. \tag{36}$$

Neither the dual $\tilde{C}$, nor the co-copula C^* of a copula C is a copula [22]; on the other hand, the survival copula $\hat{C}$ associated to C is a copula.

Using this terminology, the dual lower bound function L_C can be written compactly as

$$\begin{aligned} L_C(\alpha, \beta, \gamma) &= 1 - U_C(1 - \gamma, 1 - \beta, 1 - \alpha) \\ &= 1 - \tilde{C}(1 - \gamma, 1 - \beta) = \hat{C}(\gamma, \beta)\,. \end{aligned}$$

4.4 The case of t-norms

Corollary 2 applies in particular to t-norms that are copulas as well. Many parametric families of t-norms contain a subfamily of copulas [21]. On the other hand, there also exist lists of parametric families of copulas, most of them containing a parametric subfamily of t-norms [25].

One of the most important parametric t-norm families is the Frank family $(T_\lambda^{\mathbf{F}})_{\lambda \in [0,\infty]}$ [15], which turns out to be also a family of copulas. For $\lambda \in\,]0,1[\cup]1,\infty[$, the Frank t-norm $T_\lambda^{\mathbf{F}}$ is defined by

$$T_\lambda^{\mathbf{F}}(x, y) = \log_\lambda \left(1 + \frac{(\lambda^x - 1)(\lambda^y - 1)}{\lambda - 1}\right)\,. \tag{37}$$

As limit cases, one obtains $T_{\mathbf{M}}$ $(\lambda \to 0)$, $T_{\mathbf{P}}$ $(\lambda \to 1)$ and $T_{\mathbf{L}}$ $(\lambda \to \infty)$.

Although they appear to be quite technical, the Frank t-norms are important solutions of an often encountered functional equation. To that end, we first recall the notion of an ordinal sum of t-norms [21].

Proposition 5 *Consider a countable family* $(T_\alpha)_{\alpha\in A}$ *of t-norms and a corresponding family* $(]a_\alpha, e_\alpha[)_{\alpha\in A}$ *of non-empty, pairwise disjoint open subintervals of* $[0,1]$*. The binary operation* T *on* $[0,1]$ *defined by*

$$T(x,y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha) T_\alpha \left(\dfrac{x-a_\alpha}{e_\alpha - a_\alpha}, \dfrac{y-a_\alpha}{e_\alpha - a_\alpha} \right) & \text{, if } (x,y) \in [a_\alpha, e_\alpha]^2 \,, \\ \min(x,y) & \text{, otherwise.} \end{cases}$$

is a t-norm, and is called the ordinal sum of the summands $\langle a_\alpha, e_\alpha, T_\alpha \rangle$*,* $\alpha \in A$*.*

Ordinal sums of Frank t-norms were shown to be the only t-norms T solving the functional equation

$$T(x,y) + S(x,y) = x + y$$

for some t-conorm S. In particular, when $T = T_\lambda^{\mathbf{F}}$ this t-conorm is nothing else but the Frank t-conorm $S_\lambda^{\mathbf{F}}$ which coincides with the dual t-conorm of $T_\lambda^{\mathbf{F}}$ in the sense of (4):

$$S_\lambda^{\mathbf{F}}(x,y) = 1 - T_\lambda^{\mathbf{F}}(1-x, 1-y) \,. \tag{38}$$

In the latter case, Corollary 2 can be rephrased as follows.

Proposition 6 *A reciprocal relation* Q *on* A *is* $T_\lambda^{\mathbf{F}}$*-transitive if and only if it is cycle-transitive w.r.t. the upper bound function* $U_\lambda^{\mathbf{F}}$ *defined by*

$$U_\lambda^{\mathbf{F}}(\alpha,\beta,\gamma) = S_\lambda^{\mathbf{F}}(\alpha,\beta) \,. \tag{39}$$

Note that due to (38), the dual lower bound function $L_\lambda^{\mathbf{F}}$ is given by

$$L_\lambda^{\mathbf{F}}(\alpha,\beta,\gamma) = T_\lambda^{\mathbf{F}}(\beta,\gamma) \,.$$

From Proposition 6 we obtain the following special cases.

(a) As mentioned twice before, a reciprocal relation Q is $T_{\mathbf{M}}$-transitive if and only if it is cycle-transitive w.r.t. the self-dual upper bound function $U_{\mathbf{M}}$ defined by

$$U_{\mathbf{M}}(\alpha,\beta,\gamma) = \max(\alpha,\beta) = \beta \,. \tag{40}$$

Hence, for a $T_{\mathbf{M}}$-transitive reciprocal relation Q it must hold that $\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 = \beta_{abc}$, or equivalently, $\alpha_{abc} + \gamma_{abc} = 1$, for any $(a,b,c) \in A^3$.

(b) As proven in detail, a reciprocal relation Q is $T_{\mathbf{P}}$-transitive if and only if it is cycle-transitive w.r.t. the upper bound function $U_{\mathbf{P}}$ defined by

$$U_{\mathbf{P}}(\alpha,\beta,\gamma) = \alpha + \beta - \alpha\beta \,. \tag{41}$$

(c) A reciprocal relation Q is $T_{\mathbf{L}}$-transitive if and only if it is cycle-transitive w.r.t. the upper bound function $U_{\mathbf{L}}$ defined by

$$U_{\mathbf{L}}(\alpha,\beta,\gamma) = \min(\alpha+\beta,1)\,.$$

Hence, for a $T_{\mathbf{L}}$-transitive reciprocal relation Q it must hold that $\alpha_{abc} + \beta_{abc}+\gamma_{abc}-1 \leq \min(\alpha_{abc}+\beta_{abc},1)$, for any $(a,b,c) \in A^3$. If $\alpha_{abc}+\beta_{abc} < 1$, then this inequality is trivially fulfilled. Therefore, a reciprocal relation Q is $T_{\mathbf{L}}$-transitive if and only if it is cycle-transitive w.r.t. the simpler equivalent upper bound function $U'_{\mathbf{L}}$ defined by

$$U'_{\mathbf{L}}(\alpha,\beta,\gamma) = 1\,. \tag{42}$$

Note that the same equivalence holds for the less elegant equivalent upper bound function $U''_{\mathbf{L}}$ defined by

$$U''_{\mathbf{L}}(\alpha,\beta,\gamma) = \begin{cases} 1 & \text{, if } \alpha+\beta \geq 1\,, \\ 2 & \text{, if } \alpha+\beta < 1\,. \end{cases} \tag{43}$$

Expressions (40)–(42) nicely illustrate that $T_{\mathbf{M}}$-transitivity implies $T_{\mathbf{P}}$-transitivity and that $T_{\mathbf{P}}$-transitivity implies $T_{\mathbf{L}}$-transitivity.

5 Stochastic transitivity as cycle-transitivity

5.1 Stochastic transitivity

In this section, we propose a general notion of stochastic transitivity and show when and how it fits into the framework of cycle-transitivity.

Definition 8 *Let g be an increasing $[1/2,1]^2 \to [0,1]$ mapping. A reciprocal relation Q on A is called g-stochastic transitive if for any $(a,b,c) \in A^3$ it holds that*

$$(Q(a,b) \geq 1/2 \;\wedge\; Q(b,c) \geq 1/2) \;\Rightarrow\; Q(a,c) \geq g(Q(a,b),Q(b,c))\,. \tag{44}$$

This definition includes many well-known types of stochastic transitivity. Indeed, g-stochastic transitivity is known as

(i) strong stochastic transitivity when $g = \max$ [24];
(ii) moderate stochastic transitivity when $g = \min$ [24];
(iii) weak stochastic transitivity when $g = 1/2$ [24];
(iv) λ-transitivity, with $\lambda \in [0,1]$, when $g = \lambda \max + (1-\lambda)\min$ [2].

It is clear that strong stochastic transitivity implies λ-transitivity, which implies moderate stochastic transitivity, which, in turn, implies weak stochastic transitivity.

5.2 Stochastic transitivity as cycle-transitivity

Proposition 7 *Let g be a commutative, increasing $[1/2,1]^2 \to [0,1]$ mapping such that $g(1/2,x) \leq x$ for any $x \in [1/2,1]$. A reciprocal relation Q on A is g-stochastic transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_g defined by*

$$U_g(\alpha,\beta,\gamma) = \begin{cases} \beta+\gamma-g(\beta,\gamma) & , \textit{ if } \beta \geq 1/2 \wedge \alpha < 1/2\,, \\ \min(\alpha+\beta-g(\alpha,\beta), \beta+\gamma-g(\beta,\gamma), \\ \qquad\qquad \gamma+\alpha-g(\gamma,\alpha)) & , \textit{ if } \alpha \geq 1/2 \\ 2 & , \textit{ if } \beta < 1/2\,. \end{cases} \tag{45}$$

Proof First of all, again a tedious, yet simple verification shows that for a function g with the given properties the corresponding function U_g defined in (45) belongs to $\mathcal{U}$. Essential is the additional condition $g(1/2,x) \leq x$.

Consider a reciprocal relation Q on A and $(a,b,c) \in A^3$. If $\beta_{abc} \geq 1/2$, then also $\gamma_{abc} \geq 1/2$ and at least two of the three elements q_{ab}, q_{bc} and q_{ac} are greater than or equal to $1/2$. In this case, g-stochastic transitivity requires that $1-\alpha_{abc} \geq g(\beta_{abc},\gamma_{abc})$. If $\alpha_{abc} < 1/2$, this inequality is the only one that must hold for (a,b,c) (and cyclic permutations of it) and g-stochastic transitivity turns out to be equivalent to the condition:

$$\alpha_{abc}+\beta_{abc}+\gamma_{abc}-1 \leq \beta_{abc}+\gamma_{abc}-g(\beta_{abc},\gamma_{abc})\,.$$

However, if $\alpha_{abc} \geq 1/2$, then two more inequalities must hold, namely $1-\gamma_{abc} \geq g(\alpha_{abc},\beta_{abc})$ and $1-\beta_{abc} \geq g(\alpha_{abc},\gamma_{abc})$, and the three inequalities together yield the condition $\alpha_{abc}+\beta_{abc}+\gamma_{abc}-1 \leq \min(\alpha_{abc}+\beta_{abc}-g(\alpha_{abc},\beta_{abc}), \beta_{abc}+\gamma_{abc}-g(\beta_{abc},\gamma_{abc}), \gamma_{abc}+\alpha_{abc}-g(\gamma_{abc},\alpha_{abc}))$. If $\beta_{abc} < 1/2$, there is no upper bound for $\alpha_{abc}+\beta_{abc}+\gamma_{abc}-1$, which means that we can just put 2. Summarizing, we have shown that g-stochastic transitivity can be reformulated as cycle-transitivity w.r.t. the upper bound function U_g. □

Note that in general the additional conditions (19) are not satisfied by an upper bound function of type (45). This is only the case when $g(1/2,1/2) > 0$ or $g(1/2,1) > 1/2$.

As in the case of fuzzy transitivity, we will establish sufficient conditions on the function g which allow to bring the upper bound function U_g in a simpler form. A first proposition restricts the range of g to the interval $[1/2,1]$. Cycle-transitivity w.r.t. U_g then always implies weak stochastic transitivity. Also, the additional conditions (19) are then trivially fulfilled.

Proposition 8 *Let g be a commutative, increasing $[1/2,1]^2 \to [1/2,1]$ mapping such that $g(1/2,x) \leq x$ for any $x \in [1/2,1]$. A reciprocal relation Q on A is g-stochastic transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_g defined by*

$$U_g(\alpha,\beta,\gamma) = \begin{cases} \beta+\gamma-g(\beta,\gamma) & , \textit{if } \beta \geq 1/2 \wedge \alpha < 1/2\,, \\ 1/2 & , \textit{if } \alpha \geq 1/2\,, \\ 2 & , \textit{if } \beta < 1/2\,. \end{cases} \tag{46}$$

Proof Consider a reciprocal relation Q on A and $(a,b,c) \in A^3$. In view of Proposition 7, we only need to consider the case $\alpha_{abc} \geq 1/2$ and we know that in this case g-stochastic transitivity requires that $1-\alpha_{abc} \geq g(\beta_{abc},\gamma_{abc})$, $1-\beta_{abc} \geq g(\alpha_{abc},\gamma_{abc})$ and $1-\gamma_{abc} \geq g(\alpha_{abc},\beta_{abc})$. Since g takes values in $[1/2,1]$, this can only hold if $\alpha_{abc} \leq 1/2$, $\beta_{abc} \leq 1/2$ and $\gamma_{abc} \leq 1/2$. Since $\alpha_{abc} \geq 1/2$ it then follows that $\alpha_{abc} = \beta_{abc} = \gamma_{abc} = 1/2$. An equivalent way of arriving at this single possibility is by requiring that $\alpha_{abc}+\beta_{abc}+\gamma_{abc}-1 \leq 1/2$ in case $\alpha_{abc} \geq 1/2$. □

From Proposition 8 we obtain as special cases:

(a) A reciprocal relation Q is strongly stochastic transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_{ss} defined by

$$U_{ss}(\alpha,\beta,\gamma) = \begin{cases} \beta & , \text{if } \beta \geq 1/2 \wedge \alpha < 1/2\,, \\ 1/2 & , \text{if } \alpha \geq 1/2\,, \\ 2 & , \text{if } \beta < 1/2\,. \end{cases} \tag{47}$$

(b) A reciprocal relation Q is moderately stochastic transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_{ms} defined by

$$U_{ms}(\alpha,\beta,\gamma) = \begin{cases} \gamma & , \text{if } \beta \geq 1/2 \wedge \alpha < 1/2\,, \\ 1/2 & , \text{if } \alpha \geq 1/2\,, \\ 2 & , \text{if } \beta < 1/2\,. \end{cases} \tag{48}$$

(c) A reciprocal relation Q is weakly stochastic transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_{ws} defined by

$$U_{ws}(\alpha,\beta,\gamma) = \begin{cases} \beta+\gamma-1/2 & , \text{if } \beta \geq 1/2 \wedge \alpha < 1/2\,, \\ 1/2 & , \text{if } \alpha \geq 1/2\,, \\ 2 & , \text{if } \beta < 1/2\,. \end{cases} \tag{49}$$

(d) A reciprocal relation Q is λ-transitive, with $\lambda \in [0,1]$, if and only if it is cycle-transitive w.r.t. the upper bound function U_λ defined by

$$U_\lambda(\alpha,\beta,\gamma) = \begin{cases} \lambda\beta+(1-\lambda)\gamma & , \text{if } \beta \geq 1/2 \wedge \alpha < 1/2\,, \\ 1/2 & , \text{if } \alpha \geq 1/2\,, \\ 2 & , \text{if } \beta < 1/2\,. \end{cases} \tag{50}$$

A final simplification, eliminating the special case $\alpha = 1/2$ in (46), is obtained by requiring g to have as neutral element $1/2$, i.e. $g(1/2, x) = g(x, 1/2) = x$ for any $x \in [1/2, 1]$.

Proposition 9 *Let g be a commutative, increasing $[1/2, 1]^2 \to [1/2, 1]$ mapping with neutral element $1/2$. A reciprocal relation Q on A is g-stochastic transitive if and only if it is cycle-transitive w.r.t. the upper bound U_g defined by*

$$U_g(\alpha, \beta, \gamma) = \begin{cases} \beta + \gamma - g(\beta, \gamma) & , \text{ if } \beta \geq 1/2\,, \\ 2 & , \text{ if } \beta < 1/2\,. \end{cases} \tag{51}$$

Proof Consider a reciprocal relation Q on A and $(a, b, c) \in A^3$. As in the proof of Proposition 8, we only need to consider the case $\alpha_{abc} \geq 1/2$ in which g-stochastic transitivity is equivalent to $\alpha_{abc} = \beta_{abc} = \gamma_{abc} = 1/2$. We need to show that an equivalent way of arriving at this single possibility, knowing that g has neutral element $1/2$, is by requiring in this case that $\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \leq \beta_{abc} + \gamma_{abc} - g(\beta_{abc}, \gamma_{abc})$, or equivalently, $1 - \alpha_{abc} \geq g(\beta_{abc}, \gamma_{abc})$. Indeed, since g has neutral element $1/2$, it holds that $g \geq \max$, and we must have that $1 - \alpha_{abc} \geq \gamma_{abc}$, which, given $\alpha_{abc} \geq 1/2$, only occurs when $\alpha_{abc} = \gamma_{abc} = 1/2$, whence also $\beta_{abc} = 1/2$. □

This proposition implies in particular that strong stochastic transitivity ($g = \max$) is equivalent to cycle-transitivity w.r.t. the simplified upper bound function U'_{ss} defined by

$$U'_{ss}(\alpha, \beta, \gamma) = \begin{cases} \beta & , \text{ if } \beta \geq 1/2\,, \\ 2 & , \text{ if } \beta < 1/2\,. \end{cases} \tag{52}$$

Note that g-stochastic transitivity w.r.t. a function $g \geq \max$ always implies strong stochastic transitivity. This means that any reciprocal relation that is cycle-transitive w.r.t. an upper bound function U_g of the form (51) is at least strongly stochastic transitive.

Comparing (40) and (42) with (52) and (48), it is clear that $T_{\mathbf{M}}$-transitivity implies strong stochastic transitivity and that moderate stochastic transitivity implies $T_{\mathbf{L}}$-transitivity.

5.3 Strong stochastic transitivity revisited

The purpose of this subsection is to show that strong stochastic transitivity can also be cast in the framework of fuzzy transitivity. To that aim, we consider the ordinal sum $T_{ss} = \{([0, \frac{1}{2}], T_{\mathbf{D}}), ([\frac{1}{2}, 1], T_{\mathbf{M}})\}$, which is given explicitly by

$$T_{ss}(x, y) = \begin{cases} \min(x, y) & , \text{ if } \max(x, y) \geq 1/2\,, \\ 0 & , \text{ if } \max(x, y) < 1/2\,. \end{cases} \tag{53}$$

This t-norm is not continuous, and therefore surely not a copula. Note that it is situated between $T_{\mathbf{L}}$ and $T_{\mathbf{M}}$.

According to Proposition 2, T_{ss}-transitivity is equivalent to cycle-transitivity w.r.t. the upper bound function $U_{T_{ss}}$ (obtained by putting $f = T_{ss}$ in (29)), of which a simplified expression is given by

$$U_{T_{ss}}(\alpha,\beta,\gamma) = \begin{cases} \beta & , \text{if } \beta \geq 1/2\,, \\ \min(\alpha+\beta,\gamma) & , \text{if } \beta < 1/2 \leq \gamma\,, \\ \alpha+\beta & , \text{if } \gamma < 1/2\,. \end{cases}$$

However, one easily verifies that $\alpha+\beta+\gamma-1 \leq \min(\alpha+\beta,\gamma)$ is trivially fulfilled when $\beta < 1/2 \leq \gamma$ and $\gamma < 1/2$. The inequality $\alpha+\beta+\gamma-1 \leq \alpha+\beta$ is even always true. Hence, an equivalent upper bound function is given by

$$U'_{T_{ss}}(\alpha,\beta,\gamma) = \begin{cases} \beta & , \text{if } \beta \geq 1/2\,, \\ 2 & , \text{if } \beta < 1/2\,. \end{cases} \tag{54}$$

Comparing (52) and (54) shows that, for reciprocal relations, strong stochastic transitivity is equivalent to T-transitivity with respect to the non-continuous t-norm T_{ss}.

5.4 Isostochastic transitivity

In Subsection 3.4, we have derived the most general polynomial self-dual upper bound functions. An alternative way of defining a family of self-dual upper bound functions goes as follows.

Proposition 10 *Let g be a commutative, increasing $[1/2,1]^2 \to [1/2,1]$ mapping with neutral element $1/2$. It then holds that any $\Delta \to \mathbb{R}$ function U of the form*

$$U_g^s(\alpha,\beta,\gamma) = \begin{cases} \beta+\gamma-g(\beta,\gamma) & , \textit{if } \beta \geq 1/2\,, \\ \alpha+\beta-1+g(1-\beta,1-\alpha) & , \textit{if } \beta < 1/2\,, \end{cases} \tag{55}$$

is a self-dual member of $\mathcal{U}$.

Proof When $\beta > 1/2$, it easily follows that the dual lower bound function $L(\alpha,\beta,\gamma)$ equals $\beta+\gamma-g(\beta,\gamma)$, and coincides with the upper bound function. When $\beta = 1/2$, both functions coincide provided that the equality

$$1/2+\gamma-g(1/2,\gamma) = \alpha-1/2+g(1/2,1-\alpha)$$

holds for any $\alpha \leq 1/2$ and $\gamma \geq 1/2$. This follows from the fact that $1/2$ is the neutral element of g. Finally, it should hold that $U(0,0,1) = 0$ and $U(0,1,1) = 1$. This is guaranteed by the fact that 1 is the absorbing element of g. Indeed, $g(x,1) \geq g(1/2,1) = 1$, and hence $g(x,1) = 1$. This concludes the proof that U is a self-dual member of $\mathcal{U}$. □

Note that the function g in Proposition 10 has the same properties as the function g in Proposition 9. Of course, the upper bound function U_g^s also satisfies the additional conditions (19). Furthermore, it is immediately clear that cycle-transitivity w.r.t. an upper bound function of the form U_g^s always implies strong stochastic transitivity.

Many of the polynomial self-dual upper bound functions can be recast in the form (55). For instance, the self-dual upper bound function $U_{\mathbf{M}}$ (which characterizes $T_{\mathbf{M}}$-transitivity) is of the form (55) with $g = \max$. As a second example, let us reconsider the case of the self-dual upper bound function $U_E(\alpha,\beta,\gamma) = \alpha\beta + \alpha\gamma + \beta\gamma - 2\alpha\beta\gamma$. Similarly as in Remark 1, solving α (resp. γ) from the equation $\alpha + \beta + \gamma - 1 = \alpha\beta + \alpha\gamma + \beta\gamma - 2\alpha\beta\gamma$ and substituting the solution in the expression for $U_E(\alpha,\beta,\gamma)$ in case $\beta \geq 1/2$ (resp. $\beta \leq 1/2$), we obtain the equivalent self-dual upper bound function

$$U'_E(\alpha,\beta,\gamma) = \begin{cases} \beta + \gamma - \dfrac{\beta\gamma}{\beta\gamma + (1-\beta)(1-\gamma)} & \text{, if } \beta \geq 1/2\,, \\ \alpha + \beta - 1 + \dfrac{(1-\alpha)(1-\beta)}{\alpha\beta + (1-\alpha)(1-\beta)} & \text{, if } \beta < 1/2\,, \end{cases} \tag{56}$$

which is of the form (55) with g defined by

$$g(x,y) = \frac{x\,y}{x\,y + (1-x)(1-y)}\,. \tag{57}$$

As self-dual upper bound functions typically turn inequalities into equalities, the following proposition does not come as a surprise. It shows that cycle-transitivity w.r.t. an upper bound function of type (55) can be seen as a variant of g-stochastic transitivity. The proof is similar to that of Propositions (7)–(9).

Proposition 11 *A reciprocal relation Q on A is cycle-transitive w.r.t. a self-dual upper bound function of type U_g^s if and only if for any $(a,b,c) \in A^3$ it holds that*

$$(Q(a,b) \geq 1/2 \ \wedge\ Q(b,c) \geq 1/2) \ \Rightarrow\ Q(a,c) = g(Q(a,b),Q(b,c))\,. \tag{58}$$

The reciprocal relation Q will also be called isostochastic transitive w.r.t. *g, or shortly, g-*isostochastic transitive.

In particular, a reciprocal relation Q is $T_{\mathbf{M}}$-transitive if and only if

$$(Q(a,b) \geq 1/2 \ \wedge\ Q(b,c) \geq 1/2) \ \Rightarrow\ Q(a,c) = \max(Q(a,b),Q(b,c))\,,$$

for any $(a,b,c) \in A^3$. Note that this is formally the same as (9) with the difference that in the latter case Q was only $\{0,1/2,1\}$-valued.

Note that the properties imposed on g in Propositions 9 and 10 are very close to the defining properties of t-conorms. Indeed, although associativity is not explicitly required, it follows quite naturally. Consider for instance a g-isostochastic transitive reciprocal relation Q such that $Q(a,b) \geq 1/2$, $Q(b,c) \geq 1/2$ and $Q(c,d) \geq 1/2$. Then it holds that

$$Q(a,d) = g(Q(a,b), Q(b,d)) = g(Q(a,b), g(Q(b,c), Q(c,d)))$$

and

$$Q(a,d) = g(Q(a,c), Q(c,d)) = g(g(Q(a,b), Q(b,c)), Q(c,d)),$$

whence at least for the triplet $(Q(a,b), Q(b,c), Q(c,d))$ the function g is associative. Adding (full) associativity makes g into a t-conorm on $[1/2,1]$, or after appropriate rescaling, into a usual t-conorm on $[0,1]$.

Proposition 12 *If g is a commutative, associative, increasing $[1/2,1]^2 \to [1/2,1]$ mapping with neutral element $1/2$, then the $[0,1]^2 \to [0,1]$ mapping S_g defined by*

$$S_g(x,y) = 2g\left(\frac{1+x}{2}, \frac{1+y}{2}\right) - 1$$

is a t-conorm.

Proof One easily verifies that since g is increasing, associative and commutative, also S_g is increasing, associative and commutative. Furthermore, S_g has 0 as neutral element since

$$S_g(0,x) = 2g(1/2, (1+x)/2) - 1 = (1+x) - 1 = x,$$

for any $x \in [0,1]$. □

The two examples of self-dual upper bound functions given above fall in the latter category. For the self-dual upper bound function U'_E in (56), the associated t-conorm S_E is given by

$$S_E(x,y) = \frac{x+y}{1+xy}, \tag{59}$$

which belongs to the parametric Hamacher t-conorm family, and is the co-copula of the Hamacher t-norm with parameter value 2 [21].

6 Conclusion

In this paper, we have presented a new and general framework for studying the transitivity of reciprocal relations. The key feature is the unorthodox evaluation: triangles are visited in a cyclic manner, while ordering the weights encountered. We have shown how various types of fuzzy and stochastic transitivity can be generalized and cast into the new framework. Interesting connections have been laid bare, while new types of transitivity have been proposed, in particular those related to self-dual upper bound functions. Operators common to both fields of fuzzy logic and probabilistic metric spaces, such as t-norms, t-conorms, commutative quasi-copulas, etc., have come to the front naturally.

In upcoming work, we will describe in detail the role of the cycle-transitivity concept in the pairwise comparison of random variables, establishing a variety of alternatives to stochastic dominance. In that case, upper bound functions of the type $U(\alpha,\beta,\gamma) = \beta+\gamma-T^{\mathbf{F}}_{\lambda}(\beta,\gamma)$, which are neither an instance of fuzzy transitivity, nor of stochastic transitivity, will turn out to play a prominent role. Note that these upper bound functions do not satisfy the last condition in (19).

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